# Planning Algorithms for a Class of Heterogeneous Multi-Vehicle Systems. ⋆

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**Abstract:** In this work we focuss on path and mission planning problems in scenarios in which two different vehicles with complementary capabilities are cooperatively employed so as to perform a desired task in an optimal way. In particular, we consider the case in which a vehicle carrier, typically slow but with virtually infinite operativity range, and a carried vehicle, which on the contrary is typically fast but with a shorter operative range, are coordinated to make the faster vehicle visit in the shortest time a certain collection of points. First, we will face a planning problem in which the order of the points to be visited is given a priori according to prescribed takeoff/landing conditions. Such a problem will be solved in an exact way by means of convex optimization arguments. Such a result will be the starting point to address a more complex mission planning problem, a Traveling Salesman Problem (TSP), in which the optimal visiting sequence of points has to be determined. For this latter problem a sub-optimal heuristic will be presented and its properties pointed out.

## 1. INTRODUCTION

The complexity of many applications envisioned for future autonomous vehicle networks, ranging from planetary exploration to rescue missions, requires a broad range of capabilities for individual units—ranging from air, ground or sea mobility, to sophisticated multi-modal sensor suites and actuation devices—which cannot be implemented on a single platform class. Rather, it may be necessary to coordinate several specialized units to attain complex objectives in a reliable, timely, and efficient fashion Murray [2007]. While considerable progresses have been made on cooperative control of networks of homogeneous vehicles (see for example Chandler et al. [2002], Beard et al. [2002], Darbha [2005], Gil et al. [2003], heterogeneous networks are still relatively poorly understood. In such a direction recent developments aiming at spreading the adoption of unmanned systems in real-world operational scenarios - Parker [2008], Murphy et al. [2008] - consist of the employment of cooperating mobile robots (Cao et al. [1997]), often denoted as multiple mobile robot systems, combining the characteristics of heterogeneous vehicles with complementary features. To understand how to optimally exploit the different capabilities of each individual unit and obtain the desired final behavior, the team is required to be suitably coordinated through advanced planning and control algorithms.

One of the most challenging scenarios which future multiple autonomous vehicle systems will be required to face - Murphy [2004] - is represented by rescue missions, where such systems will be used in order to extend the range of operations and, simultaneously, reduce the risks for human rescuers. Rescue missions are particularly challenging because they can take place in really unfriendly environments under adverse climate conditions and are required to be accomplished rapidly and with a high efficiency. Moreover, they involves a potentially wide and heterogenous set of people, including ordinary civilians, typically representing the victims of the disaster, and the rescue professionals as well. As a matter of fact, rescue missions require an advanced level of cooperation among the different agents involved in the operations. Teamwork, which implies collaboration among the several units involved, plays a role of paramount importance because the agents are required to accomplish critical operations in the minimum possible time.

In this paper, we concentrate on a very simple system of heterogeneous vehicles, arising from the combination of (i) a slow autonomous surface carrier (typically a ship), with long range operational capabilities, and (ii) a faster vehicle (typically a helicopter, an UAV or an offshore vehicle) with a limited operative range. The carrier is able to transport the faster vehicle, as well as to deploy, recover, and service it. Even though this two-vehicle system is very simple, many path planning and coordination problems of interest - similar to those introduced in Frazzoli and Bullo [2004], LaValle [2006], Bertsimas and van Ryzin [1991] for other frameworks - may be defined for it. In this paper we will concentrate on a class of rescue mission planning problems assuming holonomic dynamical models to represent the behavior of both the carrier and carried vehicles. In particular, we will study and provide solutions for the so called "fast rescue problems", where the aim is at completing the missions in the "shortest possible time".

⋆ This research has been founded by the research project Giovani Ricercatori financed by the University of Calabria. Corresponding author: Roberto Naldi, CASY-DEIS University of Bologna, Via Risorgimento 2, 40136 Bologna, Italy. Tel: 0039 051 2093875, Fax: 0039 051 2093073, email: roberto.naldi@unibo.it

The paper is organized as follows. In Section 2, the carried-vehicle system and the associated notation is formally introduced. Then, in Section 3, we deal with a path planning problem consisting of visiting a prescribed sequence of points by following a pre-defined sequence of take-offs and landings. Finally, in Section 4 we consider the Traveling Salesman Problem for the proposed vehicle system. Some conclusions end the paper.

## 2. THE CARRIER-VEHICLE SYSTEM

The system we are going to deal with is composed by two different vehicles, a *vehicle carrier* (also denoted in the following as *carrier*), whose variables and functions will be denoted by subscript $\cdot_c$, and a *carried vehicle* (compactly referred to as the *vehicle*), denoted by subscript $\cdot_v$. In the following we will refer to the combined system as the *carrier-vehicle system*.

To derive a mathematical model for the system, we will consider the vehicles as points belonging to the Euclidean space $\mathbb{R}^2$. Let us fix an inertial frame $F_i = \{O_i, \overrightarrow{i}, \overrightarrow{j}\}$ and let be

$$p_c(t) = [\, x_c(t) \;\, y_c(t) \,]^T \;\; p_v(t) = [\, x_v(t) \;\, y_v(t) \,]^T \, . \qquad (1)$$

respectively the position of the carrier and of the vehicle at time $t$ in the inertial frame $F_i$. We will assume that the position of the carrier $p_c(t)$ evolves accordingly to the first order O.D.E

$$\dot{x}_c = V_c \cos(\phi_c), \; \dot{y}_c = V_c \sin(\phi_c) \qquad (2)$$

with $V_c \in \mathbb{R}^+$ the fixed velocity of the carrier and $\phi_c \in \mathbb{R}$ the control input. This implies that the class of the admissible carrier's paths are all continuous curves in the two dimensional Euclidean space which are traveled up with a velocity lower than or equal to $V_c$. In modeling the dynamics of the carried vehicle we distinguish between two different situations:

(1) when it is not carried it evolves following its free planar motion:

$$\dot{x}_v = V_v \cos(\phi_v), \; \dot{y}_v = V_v \sin(\phi_v) \qquad (3)$$

with $V_v \in \mathbb{R}^+$, $V_v > V_c$ and $\phi_v \in \mathbb{R}$ the control input for the vehicle.

(2) when it is carried, its position coincides with the carrier position, $p_v(t) = p_c(t)$.

From the above arguments it appears that the carried vehicle dynamics shows an intrinsically hybrid behavior.

Because one of the distinguish features of the carried-vehicle is to have a finite operativeness (e.g. *fuel*), we assume that, after leaving the carrier deck, it can operate as a stand-alone vehicle only for a limited time $\bar{a}$. For the sake of simplicity, it is supposed that anytime the faster vehicle comes back to the carrier its operativeness is instantaneously restored.

## 3. PATH PLANNING - ORDERED VISIT OF $N$ POINTS

Goal of this section is to determine the minimum path time for those missions in which, given an initial point $p_0$ such that $p_c(0) = p_v(0) = p_0$ and a list of $n$ points $q_{list} = [q_1, ..., q_n]$, we want the carried vehicle to visit in an ordered way those points by following, for each point $q_i$, a given sequence of *takeoff - visiting the new point- landing* prescriptions. Some solutions to particular subcases of this general problem have been provided in the preliminary work Garone et al. [2008].

Please note that, as it will be clearer soon, the assumption of a prescribed *takeoff - visiting the new point- landing* sequence is realistic in many situation (e.g. in rescue mission to bring people a.s.a.p safely on the ship) and makes the optimization problem tractable. Moreover, as it will be detailed in the sequel, this visiting sequence is the optimal one for the relevant case the points to be visited are "far enough" one to each other.

For what regarding the time to be minimized, some comments are in order. In fact, many different notions of mission completion time may be given. In this paper, we will restrict the attention to the following three cases:

**M1** the mission is completed after both the carrier and the carried vehicle reaches a certain final position $p_f$ (representing for instance the harbor, e.g. it could be that $p_f \equiv p_0$);

**M2** the mission is completed after the faster vehicle safely has visited the last point $q_n$;

**M3** the mission is complete after the carried vehicle reaches $p_f$ (eventually in an unsafe way);

where we assume a point is visited in a safe way by the carried vehicle if, after the visit, its residual operativeness allows its recovery to the ship.

The first mission completion time corresponds to the fact we want to minimize the navigation time of the carrier and may corresponds to some kind of economical optimization. The second one, instead, corresponds to the will of visiting all the $n$ points as soon as possible. It is the case, for instance, where we want to check pre-defined locations in order to detect in the shortest time possible anomalous events. The last case, instead, is typical for rescue missions in which humans lives are involved: at first the faster vehicle visits the points and eventually brings peoples on the carrier, then, with a last flight, it brings them back to an harbor as fast as possible and lands there. The above problems may be formalized as follows.

First, for what it regards the $i-$th point to be visited, the carried vehicle has to be launched at the take-off point $p_{to,i}$, it has to reach $q_i$ and then go back to the carrier (that during this visiting time can move) in a landing point $p_{l,i}$. Because of the operativeness constraints, the time $t_{p_{to,i},p_{l,i}}$ between the take-off and the landing has to be lower than or equal to $\bar{a}$ , i.e.

$$t_{p_{to,i},p_{l,i}} \leq \bar{a}. \qquad (4)$$

Obviously the time between a takeoff and a subsequent landing cannot be lower than the minimum time the ship may spend to go from $p_{to,i}$ to $p_{l,i}$ that, for the geometry of the problem, is represented by the time that the ship would spend to follow the straight line between the two points at the maximum speed, i.e.

$$\frac{1}{V_c}||p_{l,i} - p_{to,i}|| \leq t_{p_{to,i},p_{l,i}}, i = 1, \ldots, n \qquad (5)$$

For the same reason, $t_{p_{to,i},p_{l,i}}$ cannot be lower than the minimum time that the faster vehicle may spend to follow the segments $\overline{p_{to,i}q_i}$ and $\overline{q_i p_{l,i}}$:

$$\frac{1}{V_v}\left(||q_i - p_{to,i}|| + ||p_{l,i} - q_i||\right) \leq t_{p_{to,i},p_{l,i}}, i = 1, \ldots, n. \tag{6}$$

Similarly, the time $t_{p_{l,i-1},p_{to,i}}$ between the previous landing point $p_{l,i-1}$ and the new take-off point $p_{to,i}$ cannot be lower than the time the carrier would spend by following the straight line between them at the maximum speed:

$$\frac{1}{V_c}||p_{l,i-1} - p_{to,i}|| \leq t_{p_{l,i-1},p_{to,i}}, i = 1, \ldots, n \tag{7}$$

where we assume $p_{l,0} = p_0$. Finally, by following the same arguments, in the case of mission M1, we have a further path between $p_{l,n}$ and $p_f$ that will be covered by the carrier in a time $t_{p_{l,n},p_f}$ that has to satisfy

$$\frac{1}{V_c}||p_{l,n} - p_f|| \leq t_{p_{l,n},p_f}, \tag{8}$$

while in the case M3 we have two further paths: the first between the last landing point and the new fast vehicle takeoff has to be by a time $t_{p_{l,n},p_{to,n+1}}$ such that

$$\frac{1}{V_c}||p_{l,n} - p_{to,n+1}|| \leq t_{p_{l,n},p_{to,n+1}}, \tag{9}$$

and a second path, made by the carried vehicle between that take-off point and the point $p_f$, that has to be covered by a time $t_{p_{to,n+1},p_f}$ such that:

$$\frac{1}{V_v}||p_{to,n+1} - p_f|| \leq t_{p_{to,n+1},p_f}, \tag{10}$$

$$t_{p_{to,n+1},p_f} \leq \bar{a}. \tag{11}$$

By using the above arguments, we can reformulate the problems in the common framework of determining a set of points $p_{to,i}, p_{l,i}, i = 1, ..., n$ (and $p_{to,n+1}$ in the M3 case) and the opportune vehicle trajectories between them, $p_0$, the target points $q_i, i = 1, ..., n$ and $p_f$, such that the time the vehicles spend to complete each path satisfies (4)-(7) (and eventually (8) or (9)-(11) in the M1 and M3 case respectively) and their sum

$$t_{tot} = \sum_{i=1}^{n} t_{p_{l,i-1},p_{to,i}} + t_{p_{to,i},p_{l,i}} \tag{12}$$

plus eventually $t_{p_{l,n},p_f}$ (in the M1 case) or $t_{p_{l,n},p_{to,n+1}} + t_{p_{to,n+1},p_f}$ (in the M3 case) is minimized. Obviously, because we are interested in minimizing the sum of the above time intervals, the trajectories between the points have to be the fastest one. This means that:

1) the optimal trajectory of the carrier between the point $p_{l,i-1}$ and the point $p_{to,i}$ for $i = 1, ..., n$ (and between $p_{l,n}$ and $p_0$ or $p_{to,n+1}$ in the M1 and M3 case respectively) is always a straight line between the two points traveled up at the maximum carrier speed $V_c$
2) in the M3 case the optimal trajectory of the carried vehicle between $p_{to,n+1}$ and $p_f$ is always a straight line traveled at the maximum speed $V_v$
3) if
$$\frac{1}{V_v}\left(||q_i - p_{to,i}|| + ||p_{l,i} - q_i||\right) \geq \frac{1}{V_c}||p_{l,i} - p_{to,i}||$$
then the minimum time $t_{p_{to,i},p_{l,i}}$ is
$$t_{p_{to,i},p_{l,i}} = \frac{1}{V_v}\left(||q_i - p_{to,i}|| + ||p_{l,i} - q_i||\right)$$
and the trajectory of the carried vehicle is composed by the two segments $\overline{p_{to,i}q_i}$ and $\overline{q_i p_{l,i}}$, covered with speed $V_v$ while the carrier can do any possible trajectory that brings it from $p_{to,o}$ to $p_{l,i}$ in exactly $t_{p_{to,i},p_{l,i}}$. *Vice versa*, if
$$\frac{1}{V_v}\left(||q_i - p_{to,i}|| + ||p_{l,i} - q_i||\right) \leq \frac{1}{V_c}||p_{l,i} - p_{to,i}||$$
the minimum time $t_{p_{to,i},p_{l,i}}$ is
$$t_{p_{to,i},p_{l,i}} = \frac{1}{V_c}||p_{l,i} - p_{to,i}||,$$
and then the the carrier has to follow the straight line between $p_{to,i}$ to $p_{l,i}$ at its maximum speed, while the carried vehicle can do any possible trajectory starting from $p_{to,o}$ that touches $q_i$ and brings the vehicle in $p_{l,i}$ in the temporal slot $t_{p_{to,i},p_{l,i}}$.

The latter discussion allows us to conclude that, to define the optimal path, all the trajectories between the starting, the take-off, the landing positions and the points to be visited have to be straight lines followed at the maximum speed with the only exception of the routes between the points $p_{to,i}$ and $p_{l,i}$ where one of the two vehicles will be allowed to follow any feasible trajectory that allows it to arrive on the *rendezvous* point at the time $t_{p_{to,i},p_{l,i}}$ defined by the time the other vehicle employs to cover its straight lines path. Therefore, the introduced path planning problems reduce to the problems of determining the set of takeoff and landing points that solves an optimization problems which, e.g. for the M2 case, is given by

$$\begin{array}{l} \min t_{tot} \\ \textit{subject to} \\ (4) - (7) \end{array} \tag{13}$$

where $t_{tot}$ is the time defined in (12). Problem M1 formulation would consist of adding the further constraint (8) and by minimizing $t_{tot} + t_{p_{l,n},p_f}$ while problem M3 would arise by adding the constraints (9)-(11) and minimizing $t_{tot} + t_{p_{l,n},p_{to,n+1}} + t_{p_{to,n+1},p_f}$. Interestingly enough, the following result can be proved:

**Lemma 1 -** Problem (13) is a convex optimization problem

*Proof:*

Being the objective function sum of variables, it is linear and then convex. Moreover, for what it regards the admissible region, linear constraints (4) and distance constraints (5) and (7) are trivially convex. It can also proved that kind of constraints (6) are convex as well, constraining sum of distances. Finally, let us consider two triple $(p'_{to}, p'_l, t')$ and $(p''_{to}, p''_l, t'')$ that satisfy

$$\frac{1}{V_v}\left(||q - p'_{to}|| + ||p'_l - q||\right) \leq t'$$
$$\frac{1}{V_v}\left(||q - p''_{to}|| + ||p''_l - q||\right) \leq t''$$

In order to prove convexity we need to prove that the triplet $(\lambda p'_{to} + (1-\lambda)p''_{to}, \lambda p'_l + (1-\lambda)p''_l, \lambda t' + (1-\lambda)t'')$ with $\lambda \in [0, 1]$ satisfies the same constraints that the two triples do singularly, which can be proved by simple triangular inequality arguments as follows

$$\frac{1}{V_v}(||q-\lambda p'_{to}+(1-\lambda)p''_{to}||+ \\ +||\lambda p'_l+(1-\lambda)p''_l-q||)= \\ =\frac{1}{V_v}(||\lambda(q-p'_{to})+(1-\lambda)(q-p''_{to})||)+ \\ +\frac{1}{V_v}(||\lambda(p'_l-q)+(1-\lambda)(p''_l-q)||) \\ \leq \\ \lambda\frac{1}{V_v}(||q-p'_{to}||+||p'_l-q||)+ \\ +(1-\lambda)\frac{1}{V_v}(||q-p''_{to}||+||p''_l-q||) \\ \leq \\ \lambda t'+(1-\lambda)t''$$

Note that the convexity of the problem allows one to obtain the optimal trajectory amongst $n$ points in a polynomial time w.r.t. the problem dimension. The latter result obviously holds true even for the modified optimization problem formulated to solve M1 and M3.

**Remark 1 -** Please note that here the employed notion of "visit of a point" $q_i$ is that the carried vehicle has to exactly arrive over $q_i$. In many applications different notions may be used. One of the most interesting is the case in which it is enough for the vehicle to arrive within a radius $r_i$ from the point $q_i$. The formalization of such a case can be covered by the presented discussions by simply modifying constraints (6) as follows :

$$\frac{1}{V_v}(||p_{r,i}-p_{to,i}||+||p_{l,i}-p_{r,i}||)\leq t_{p_{to,i},p_{l,i}}, \\ ||p_{r,i}-q_i||\leq r_i, \qquad i=1,...,n \tag{14}$$

where $p_{r,i}\in\mathbb{R}^2, i=1,..,n$ are further points to be determined and represent the points where the carried vehicle considers visited that points and starts the re-entry task. Please note that the overall optimization problem remains convex. Finally, it is worth to remark that several other interesting notions of "visit of a point" can be defined and, in many cases, do not compromise the convexity of the path planning optimization problem. □

*3.1 Determination of analytical upper and lower bounds to the optimal cost*

The availability of an effective optimization procedure able to compute in a reasonable time the optimal path for visiting an ordered list of $n$ points represents a fundamental starting point to deal with more general planning problems as those presented in the second part of the paper. However, it is worth to remark that when one comes to analyze the performance of the algorithms based on such a solution, beside the numerical procedure, an analytical characterization of the optimal cost should be preferable.

Unfortunately, up to our knowledge, an exact formula for the optimal cost is not known and it is probably very hard to be obtained. The goal of this subsection is to provide an analytical characterization of the optimal cost in terms of closed-form upper and lower bounds. For the sake of compactness and without loss of generality, here we limit the analysis to the M1 case (the other two cases can be treated in a similar way). Let us compactly denote the distance between two consecutive points to be visited as

$$d_{0,1}:=\|p_0-q_1\| \\ d_{i-1,1}:=\|q_{i-1}-q_i\|, i\in\{2,3,...n\} \\ d_{n,n+1}:=\|q_n-p_f\|.$$

Moreover, let us denote with $\theta_{list}:=[\theta_1,\theta_2,...\theta_n]$ the list of the $n$ angles such that $\theta_i\in[0,\pi]$, $i=1,...,n-1$, i.e. the set of the minimum amplitude angles formed by the segments that connect two consecutive points to be visited (for a graphical intuition see also Figure 1). Finally, let us denote with $\ell$ the sum of all the distances between the points of interest, i.e. $\ell=\sum_{i=1}^{n+1}d_{i-1,i}$. Note that, by construction, $\ell$ denotes the *length of the shortest path* a single holonomic vehicle should complete to visit all the points of interest. By simply noting that the carrier/vehicle systems can never be faster of a single vehicle with maximum velocity $V_v$ and unlimited operativity range, the latter allows one to define a trivial lower bound for the optimal cost as follows

$$t_L(\ell)=\frac{\ell}{V_v}. \tag{15}$$

Similarly a trivial upper bound can be determined by considering a solution in which the shortest path of length $\ell$, completed at the speed of the carrier, i.e. with the vehicle always on the carrier's deck in order to fulfill the operativeness constraints:

$$t_U(\ell)=\frac{\ell}{V_c}. \tag{16}$$

Less conservative lower-bounds for the problems can be obtained as follows. Consider the operativeness constraints. Then, the maximum speed of the combined vehicle over a certain path is obtained by considering the vehicle employed for the largest possible amount of time, which is given by $\bar{a}\,n$, where $n$ denotes the number of takeoffs for the problem M1 which is equal to the cardinality of the set $q_{list}$. It follows that a more tight lower bound is given by

$$t_L(\ell,n)=\max\left\{\left(t_U(\ell)-\sum_{i=1}^{n}\frac{V_v\bar{a}}{V_c}+\sum_{i=1}^{n}\bar{a}\right),t_L(\ell)\right\}. \tag{17}$$

In practice, for each point which can be reached only by the faster vehicle, the lower bound considers a segment of the shortest path $\ell$ of length $V_v\bar{a}$ completed at the maximum speed $V_v$.

For what it regards the determination of a tighter upper bound, first let us assume that the distances $d_{i-1,i}, i=1,...,n+1$ are always greater than or equal to the maximum operative radius of the vehicle i.e.

$$d_{i-1,i}\geq\bar{a}V_v,\ \forall i=1,...,n+1. \tag{18}$$

Under this assumption an upper-bound to problem M1 is obtained by building up the feasible solution depicted in Figure 1. The idea is to determine, for $i=1,...,n$, two points $p_{to,i},p_{l,i}$ equidistant from $q_i$ and lying over the segments $\overline{q_{i-1},q_i}$ and $\overline{q_i,q_{i+1}}$ respectively, such that constraints (4)-(6) are satisfied. Interestingly enough, this construction allows one to directly achieve an expression for the distance between $q_i$ and both the points. In fact, by means of the triangle properties we obtain that

$$||p_{to,i}-q_i||=||p_{l,i}-q_i||= \\ =\begin{cases}(V_v\bar{a})/(2) & \text{if } \theta_i\leq 2\arcsin(V_c/V_v)\\ (V_v-V_c)/(2sin(\theta_i/2)) & \text{else}\end{cases}$$

The latter allows one to obtain the following cost:

$$t_U(\ell, n, \theta_{list}) = t_L(\ell, n) + \sum_{i=1}^{n} \frac{\Delta(\theta_i)}{V_c} \leq t_U(\ell) \qquad (19)$$

with

$$\Delta(\theta_i) := \begin{cases} 0 \ \text{ if } \theta \leq 2 \arcsin\left(\frac{V_c}{V_v}\right) \\ \bar{a}\left(V_v - V_c \frac{1}{\sin \theta/2}\right). \end{cases} \qquad (20)$$

that is an upper-bound to the optimal problem under the assumption (18).

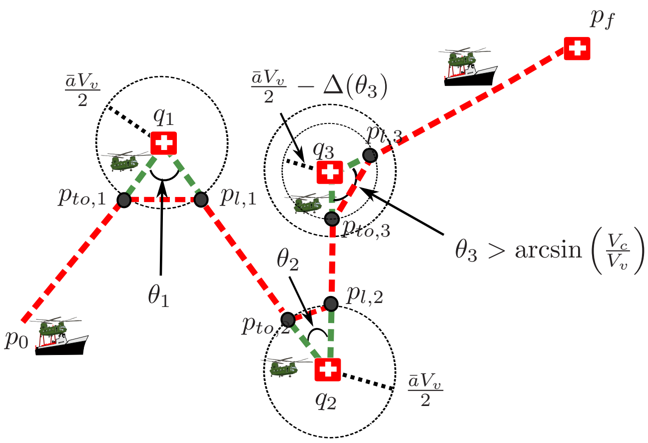


Fig. 1. The geometric interpretation behind the proposed algorithm to build a feasible solution for the problem M1 to visit a set of $n$ points (3 in the figure).

Please note that under certain conditions on the angles, i.e. $\theta_i \leq 2 \arcsin(V_c/V_v)$ for all $\theta_i \in \theta_{list}$, the proposed upper-bound precisely matches the cost of the lower bound (17), and indeed it represents (one of) the optimal solution. Moreover, by construction, the cost computed using (19) is always lower than or equal to the cost computed using (17). In particular, the two costs are equivalent only in the case in which all the points are on the same line, i.e. $\theta_i = \pi$ for all $\theta_i \in \theta_{list}$. In order to generalize (19) to the case where the points to be visited are not necessarily far each other, a simple idea is to virtually reduce the operativeness of the vehicle in order to obtain a new set of parameters compatible with the far points assumption. Namely, let us denote with $d_{min} := \min_{i=1,\ldots,n} d_{i-1,i}$ and let $\bar{a}' \leq \bar{a}$ such that

$$\bar{a}' = d_{min}/V_v$$

Then, by assuming that the vehicle operativeness equals $\bar{a}'$, we can compute the following upper-bound

$$t'_U(\ell, n, \theta_{list}) = t_L(\ell, n) + \sum_{i=1}^{n} \frac{\Delta'(\theta_i, \bar{a}, \bar{a}')}{V_c} \leq t_U(\ell) \qquad (21)$$

with

$$\Delta'(\theta, \bar{a}, \bar{a}') := \begin{cases} (\bar{a} - \bar{a}')V_v \ \text{ if } \theta \leq 2 \arcsin\left(\frac{V_c}{V_v}\right) \\ \bar{a}V_v - \frac{\bar{a}'V_c}{\sin \theta/2}. \end{cases} \qquad (22)$$

## 4. THE TRAVELING SALESMAN PROBLEM

The mission planning problem that we will study in this section is a version of the Traveling Salesman Problem (TSP) for the multi-vehicle system into consideration. For the sake of simplicity and without loss of generality we will focus on the following settings, based on the completion time M1 introduced in section 3. Let us assume that the initial positions of both vehicle and the carrier $p_c(0) = p_v(0) = p_0$ and an unordered set of $n$ points $q_{set} = \{q_1, \ldots, q_n\}$ to be visited are given. The goal is to determine the optimal trajectory for the carried vehicle that, from the initial position, touches all $n$ points and finally, to conclude a cycle, returns to the initial point $p_0$ (i.e. $p_f \equiv p_0$) for landing on the carrier vehicle.

As well known TSP problems are typical NP-Hard problems and no polynomial-time algorithms are known to solve them Papadimitriou and Steiglitz [1982]. For such a reason, in most practical cases, heuristics have to be used.

In this paper, in order to deal with the particular TSP problem in exam, hereafter denoted as Carrier/Carried-TSP (CC-TSP), we propose an heuristic algorithm based on the Euclidean TSP.

The Euclidean TSP (E-TSP) is a particular case of the general TSP in which, given $n$ points in the space, we want to determine the optimal sequence that minimizes the sum of the Euclidean distances amongst consecutive points. One of the main feature of this class of TSP problems is that, although still NP-Hard, it admits a polynomial-time approximation scheme (see Arora [1998]). This means that, for any scalar $e > 0$, it is possible to find in a polynomial time a tour whose length is at most $(1 + 1/e)$ times the optimal length. Then, in practice, for any instance of the $E-TSP$ we can obtain an *almost*-optimal solution in a reasonable time.

The CC-TSP heuristic here proposed consists of the following two steps

(1) determine the visiting order of the *almost*-optimal E-TSP tour for the set of points $\{p_0\} \cup q_{set}$;
(2) using the above visiting order, solve the convex optimization problem defined in section 3 for the M1 problem with $p_f = p_0$.

The idea behind this approach is that, as it will be clear soon, the completion time of the CC-TSP is connected to the sum of the distances between points, and then, the minimization of E-TSP leads usually to achieve a reasonably good CC-TSP solution. In particular, it is possible to prove the following Lemma:

**Lemma 4 -** Let $p_0$ and a set of $n$ points to be visited $q_{list}$ be given. Let $\ell^{opt}_{ETSP}$ be the length of the optimal E-TSP tour for the given set of points. Then, a lower bound for the CC-TSP is given by $t_L(\ell^{opt}_{ETSP}, n)$.
*Proof:*
By recalling the definition of the lower bound (17), once the number of takeoff points $n$ is fixed (according to cardinality of the set $q_{list}$ for the completion time M1), it follows that

$$\ell_1 \leq \ell_2 \Rightarrow t_L(\ell_1, n) \leq t_L(\ell_2, n). \qquad (23)$$

Let $\ell$ be the length of a generic hamiltonian cycle for the points $p_0$, $q_{list}$ starting at the point $p_0$. From the definition of E-TSP it results that $\ell^{opt}_{ETSP} \leq \ell$ and then $t_L(\ell^{opt}_{ETSP}, n) \leq t_L(\ell, n)$. This proves that the lower bound computed considering the optimal E-TSP tour is also a lower bound for the CC-TSP because no other choice of the visiting order can obtain a lower value. Moreover, by exploiting the geometrical construction proposed in subsection 3.1, it is possible to bound the committed maximum error via the proposed heuristic by simply computing the upper bound for the given sequence. The following Lemmma can be stated:

**Lemma 5 -** Let the set $\{p_0\} \cup q_{list}$ be given and $\ell_{ETSP}$ denote the length of the $(1+1/e)$-approximated optimal E-TSP tour with $e \geq 0$. Then, the completion time $t_{CC-TSP}^{heu}$ obtained with the CC-TSP heuristic has a cost which is at most $\varepsilon$ times the optimal one with $\varepsilon$ given by

$$\varepsilon := \frac{t'_U(\ell_{ETSP}, n, \theta_{list})}{t_L\left(\frac{\ell_{ETSP}}{1+1/e}\right)}$$

where $t'_U(\cdot)$ is defined in(21).

*Proof:*

Because $\ell_{ETSP}$ denotes the length of the *quasi*-optimal E-TSP tour we have that the optimal length $\ell_{ETSP}^{opt}$ is bounded from the below by a function of the scalar parameter $e$

$$\ell_{ETSP}^{opt} \geq \frac{\ell_{ETSP}}{1+1/e}.$$

Applying Lemma 4 we have that the optimal solution of the CC-TSP is then greater than or equal to

$$t_L\left(\frac{\ell_{ETSP}}{1+1/e}, n\right).$$

Moreover, by following subsection 3.1 arguments, it is possible to build up an upper-bound considering the sequence of points obtained with the *almost*-optimal E-TSP and the angles $\theta_{list}$ which correspond to this solution. The above Lemma shows some important properties of the proposed heuristic. In particular, it reveals how the distances of the points *sufficiently far* each others (satisfy (18)) and how the conditions on the angles $\theta_i$ formed by the segments connecting the points in the obtained order, which on the other hand have to be sufficiently small, effect the computation of the upper-bound. In particular, observe how the difference from the heuristic and the optimal CC-TSP solutions depend only by the scalar coefficient $e$, which is a parameter in the *almost*-optimal E-TSP algorithm. In all other cases, the heuristic obtains a cost which is at most $\epsilon$ higher than the optimal cost, where $\epsilon$ is by construction lower than or equal to $t'_U(\ell_{ETSP})/t_L\left(\frac{\ell_{ETSP}}{1+1/e}\right)$.

## 5. CONCLUSIONS

In this paper we have studied path planning problems for a class of carrier/carried vehicle systems in which a slow vehicle carrier with infinite operativity range cooperates with a carried vehicle which, on the contrary, is faster but has a limited operative range. In the first part of the paper the problem of determining the optimal path for an ordered visit of $n$ points through a prescribed takeoff/landing sequence has been studied and solved by means of convex optimization arguments. Then a Traveling Salesman Problem for this class of vehicles has been formulated. Being this problem hard to be solved in an exact way, analytical heuristic solutions have been proposed and their performances analyzed.

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