Discontinuous Control Systems in the Presence of Measurement Noise

Emanuele Garone ° Roberto Naldi † Alessandro Casavola ° Emilio Frazzoli ‡

Abstract— **In this paper the properties of a class of piecewise continuous systems subject to measurement noise are studied especially regarding their possibility to be stabilized around a whatever vector whitin a finite set of desired stabilization points. Such a problem is a typical control issue in many real systems (vehicles attitude control, robotic manipulators, etc.). Conditions under which global asymptotic stability to a set of points can be guaranteed have been studied in the past by exploiting the intrinsic "holding" nature of the discrete-time control framework, which can be emulated in continuous-time by the introduction of sample-and-hold strategies. Because such conditions can be conservative in some cases, new policies are here introduced and investigated based on the notion of "strategy holding" in time (by providing a slower switching policy sampling time) and space (by using hysteresis regions).**

I. Introduction

Piecewise continuous control laws (see [1], [8], [11]) are often employed in applications where the point in the state space to which the system has to be asymptotically stabilized is not defined univocally a priori but may depend on the current state of the system. In some applications of particular interest, such as attitude stabilization (see for example [16]), the use of discontinuous control laws is a consequence of topological issues that make it impossible to obtain global results using continuous feedback (see [2]). In this scenario the presence of noise, even arbitrarily small, may result in undesired switches between different control laws which could eventually prevent the desired control goal to be achieved at all. Different approaches have been proposed in literature to solve this class of problems. In particular, for continuous-time systems, "sampling-and-hold" control policies [13], [10] and hysteresis [12], [6] have been recently applied.

In this paper, motivated by the framework introduced in [9] and [4] and by the interest that these kind of systems have in real applications, we propose a detailed analysis of the above problem for both continuous and discrete time systems. In particular, we will show how discrete-time control formulations, under certain conditions on the form of the control law, are able to achieve implicitly some robustness properties. This is due to the implicit "one-step input holding nature" of the discrete time framework that can be "emulated" in the continuous time framework by means of sample-and-hold techniques. However, in both cases, such robustness is attained at the cost of very restrictive hypotheses on the control laws, depending on the measurement noise magnitude.

° Università della Calabria, Rende (CS), 87036, Italy
† Università di Bologna, Bologna, 40133, Italy
‡ Massachusetts Inst. of Technology, Cambridge, MA, 02139, USA

For such reason, we propose and analyze different approaches to guarantee a desired system behavior with less conservative conditions. In particular the "scheduling sample and hold" strategy is proposed. The main idea of such an approach is to allow switches only at certain sampling times. The resulting control is a continuous law scheduled with a certain sampling time in the continuous case and a multi-rate control law in which the policy is chosen every N steps in the discrete time case. One of the main advantage of such a strategy is that it introduces a decoupling property in the synthesis of the overall control. In fact, it allows one to design the control laws (almost) regardless to the effects of the measurement noise and, afterward, choose the needed sample time of the scheduling policy. Moreover we will analyze further techniques based on hysteresis on the state space. Similar strategies have been introduced in the continuous time control systems for example in [12]. Here we will study this kind of strategy by underlining the main properties in both continuous and discrete time frameworks. Finally, in order to exploit the advantages of both time-holding and state hysteresis strategies, mixed policies will be proposed and carefully analyzed.

II. Problem Statement

Consider the following dynamical system subject to measurement noise:

$$\begin{array}{l}\delta x(t) = f\left(x\left(t\right), u\left(t\right)\right),\\ y\left(t\right) = x\left(t\right) + e\left(t\right),\end{array} \tag{1}$$

where δ represents the derivatives in the continuous-time case and the one-step ahead shift operator $\delta x(t) = x(t+1)$ in the discrete time case, and:

- $x(t) \in X \subseteq \mathbb{R}^n$ is the unknown state,
- $y(t) \in X \subseteq \mathbb{R}^n$ is the output (measured state),
- $u(t) \in U \subseteq \mathbb{R}^m$ is the measurable control input,
- $e(t) \in E \subseteq \mathbb{R}^n$ is the measurement noise.

In the following, we will assume that the set E is bounded, and $e_{\max} \geq \sup_{e\in E} \|e\|$, where $\|\cdot\|$ denotes the Euclidean norm. Finally, it is supposed that the control system (1) is well-posed, i.e. :

- $\forall x(t) \in X$, $\forall \tau \geq 0$, $\forall\ u(t)|_t^{t+\tau} \in U \Rightarrow x(t+\tau) \in X$
- $\forall x(t) \in X$, $\forall u(t) \in U \Rightarrow x(t+1) \in X$.

for the continuous and discrete-time case respectively.

To introduce our main results we briefly recall the following notions of stability:

ISBN 978-963-311-369-1

Definition 1 (ISS stability with restriction [15]) *The system $\delta x = f(x(t), w(t))$ is ISS with respect to the input $w(t)$ with restriction X on the initial state and $w_{\max} > 0$ on the input if there exist a class-K function $\gamma : \mathbb{R}_{\geq} \to \mathbb{R}_{\geq}$ and class-KL function $\beta : \mathbb{R}_{\geq} \times \mathbb{R}_{\geq} \to \mathbb{R}_{\geq}$ such that*

$$\|x(t)\| \leq \beta(\|x(0)\|, t) + \gamma(\|w(t)\|) \tag{2}$$

for all $t \geq 0$, for all $w(t)$ satisfying $\limsup_{t\to\infty} \|w(t)\| \leq w_{\max}$ and for all $x(0) \in X$.

It has been proven (see for example [14], [9]) that if a system is ISS, it admits a K-asymptotic gain, i.e.

$$\limsup_{t\to\infty} \|x(t)\| \leq \gamma\left(\limsup_{k\to\infty} \|w(t)\|\right) \leq \gamma(w_{\max}).$$

□

Finally, consider the following control law

$$u(t) = \begin{cases} g_1(y(t)), & y(t) \in X_1, \\ g_2(y(t)), & y(t) \in X_2, \end{cases} \tag{3}$$

where $\{X_1, X_2\}$ is a partition of X, i.e., $X_1 \cup X_2 = X$, $X_1 \cap X_2 = \emptyset$, and g_i are continuous functions. Thereafter, $g_i(y(t))$, $i = 1, 2$, is assumed to be such that the closed loop system $\delta x(t) = f(x(t), g_i(y(t)))$ with equilibrium point $\bar{x}_i \in X_i$ is ISS with restriction S_i on the initial state, where $X_i \oplus E = \{x \in X | \exists e \in E : y = x + e \in X_i\}$.

As it will be shown in the next section, in the presence of measurement noise, control laws of the form (3) could in general yield limit cycles and (in the continuous case) well-posedness issues arise (see [8]). In particular, it is known [13] that in the continuous-time case such problems cannot be avoided by using control laws of type (3), on the contrary here we will show that it is possible, in the discrete-time case, to find conditions under which global asymptotic stabilization of the system to a "set of points" (global set-stabilization) is ensured. The goal of this paper is to point out those conditions and, afterward, propose and study alternative switching policies able to reach the same objectives, both in the continuous and the discrete time case, with less restrictive requirements.

In order to proceed to a systematical analysis the following stability issues will be addressed:

Definition 2 (Initial state dependent stabilization) *The system (1), equipped with the control law (3) such that, for $e \equiv 0$, $\bar{x}_1 \in X_1$, $\bar{x}_2 \in X_2$ are the two equilibrium points, is said to be initial state dependent stable if for any $y(0) \in X_i$ the closed loop trajectory $x(t)$ converges to the equilibrium point $\bar{x}_i$, i.e. $\limsup_{t\to\infty} \|x(t) - \overline{x}_1\| \leq \gamma_1(e_{max})$, $\forall y(0) \in X_1$, $\limsup_{t\to\infty} \|x(t) - \overline{x}_2\| \leq \gamma_2(e_{max})$, $\forall y(0) \in X_2$.* □

Definition 3 (Global set-stabilization) *The system (1), equipped with the control law (3) such that $\bar{x}_1 \in X_1$, $\bar{x}_2 \in X_2$ are the two equilibrium points, is said to be globally stable to the set of points $\{\bar{x}_1, \bar{x}_2\}$ if, $\forall x(0) \in X$, the closed loop trajectory converges indefferently to one of the equilibrium points $\bar{x}_1$ or $\bar{x}_2$, i.e. $\limsup_{t\to\infty} \|x(t) - \overline{x}_1\| \leq \gamma_1(e_{max}) \vee \limsup_{t\to\infty} \|x(t) - \overline{x}_2\| \leq \gamma_2(e_{max})$, $\forall x(0) \in X$.* □

Notation: For notational simplicity often we refer to a system in the form $\delta x(t) = f(x(t), g_i(y(t)))$ with equilibrium point $\bar{x}_i$ to be ISS with restriction on a certain set S_i. By this notation it is more precisely meant that, given the change of coordinates $x(t) \mapsto \rho_i(t) = x(t) - \bar{x}_i$, the system $\delta\rho_i(t) = f(\rho_i(t) + \overline{x}_i, g_i(\rho_i(t) + \overline{x}_i + e(t))) - \delta\overline{x}_i$ is ISS with respect to the measurement noise $e(t)$ with restriction $Q_i = \{\rho_i | \rho_i + \bar{x}_i \in S_i\}$ on the initial state and $e_{\max}$ on the input. □

III. An attitude control example

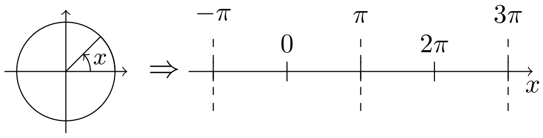

Fig. 1. Control of a rigid body rotating around a fixed axis.

Let us consider a mechanical system consisting of a rigid body rotating around a fixed axis. Let us assume that the input of the system is the angular velocity. By definition the configuration space of the system is the unit circle. Moreover, let us assume that the measurement of the angular position $y(t)$ is affected by a noise of bounded amplitude $|e| < e_{\max}$.

The overall system could be rewritten in the following way:

$$x(t+1) = x(t) + u(t), \; y(t) = x(t) + e(t), \tag{4}$$

where $x(t)$ is the angular position, and $u(t)$ represents the angular velocity.

In order to stabilize the desired equilibrium point that corresponds to $x = 2k\pi$, with $k \in Z$, we choose the following discontinuous output feedback law (see also [2]):

$$u_k(t) = -K(y(t) - 2k\pi) \;\text{ if } y(t) \in X_k, \tag{5}$$

where $X_k : x \in \Big((2k-1)\pi; (2k+1)\pi\Big]$. It is possible to prove that, for a fixed k, the control law (5) with $0 < K < 1$ is such that the overall closed loop system is ISS, with equilibrium point $2k\pi$, according to definition (2).

In the general case, even if each control law (i.e., for any fixed k) is ISS, the system could not converge to any of the points on $\mathbb{R}$ mapping the desired position we want to reach. In fact, under the right conditions, it is possible to build up a measurement error sequence such that a system with the shown control law never leaves a neighborhood of the border points $(2k+1)\pi$.

For instance, consider $K = e_{\max}/(\pi - 0.5\, e_{\max})$ with $|e(t)| \leq e_{\max}$ and consider the following sequence:

$$x(0) = \pi + 0.5\, e_{\max}, e(0) = -e_{\max}, y(0) = \pi - 0.5\, e_{\max}$$

$$\begin{aligned} x(1) &= \pi + 0.5\, e_{\max} - K(\pi - 0.5\, e_{\max}) \\ &= \pi + 0.5\, e_{\max} - \frac{e_{\max}}{\pi - 0.5\, e_{\max}} (\pi - 0.5\, e_{\max}) \\ &= \pi - 0.5\, e_{\max} \\ e(1) &= e_{\max}, \quad y(1) = \pi + 0.5\, e_{\max} \\ x(2) &= \pi - 0.5\, e_{\max} - K(\pi + 0.5\, e_{\max} - 2\pi) \\ &= \pi - 0.5\, e_{\max} + \frac{e_{\max}}{\pi - 0.5\, e_{\max}} (\pi - 0.5\, e_{\max}) \\ &= \pi + 0.5\, e_{\max} \end{aligned}$$

Because $x(2) = x(0)$, the sequence of measurement errors

$$e(k) = \begin{cases} -e_{\max} & k = 0, 2, 4, ... \\ e_{\max} & k = 1, 3, 5, ... \end{cases}$$

is such that the system keeps switching indefinitely between the two control laws without convergence to any of the two equilibrium points (this behavior is often referred to as *chattering*).

IV. Switching Laws

In this section we study the properties of switching control laws focusing on discrete time systems, considering both initial state stabilization and global set stabilization control problems.

A. Initial state dependent stabilization

To derive conditions sufficient to ensure initial state dependent stabilization, it is mandatory to proceed with some definitions.

Definition 4 (Prediction set) *Let us define $\hat{Y}(y, u, E)$ the set of all possible predictions of $y(t+1)$ given $y(t) = y$ and $u(t) = u$, i.e., $\hat{Y}(y, u, E) = \{y' \in X : y' = f(y - e_1, u) + e_2,\ e_1, e_2 \in E\}$.* □

The definition of prediction set allows one to formally state the following invariance property, which we will often use in this paper.

Definition 5 (Robust positive invariance [3]) *Let the system (1) under the control law $u(t) = g(y(t))$ be given. The set $X_p \subset X$ is said to be "robustly positively invariant" with respect to $g(y(t))$ if the following condition holds true:*

$$\hat{Y}(y, g(y), E) \subseteq X_p\,, \quad \forall y \in X_p\,. \qquad \square \tag{6}$$

In other words, the set X_p is robustly positively invariant if given $y(t) \in X_p$, then $y(t+1) = f(y(t) - e(t), g(y(t))) + e(t+1)$ belongs to X_p, for all $e(t), e(t+1) \in E$. Taking advantage of the definition of positive invariance, it is possible to find sufficient conditions ensuring the desired properties. The key point is that, if any of the sets X_i over which law (3) is defined is a robust positively invariant set for $g_i(y(t))$ then the laws never switch and the overall system converges to $\bar{x}_i$. Such an idea is formalized in the following theorem.

Theorem 1 (No-trespassing sufficient condition) *Let the system (1) and a piecewise continuous control law in the form (3) be given. Moreover, let $g_i(y(t))$ be such that the closed-loop system $x(t+1) = f(x(t), g_i(y(t)))$ with equilibrium point $\overline{x}_i$ is ISS with restriction S_i on the initial state, with $X_i \subseteq X_i \oplus E \subseteq S_i$, and $\bar{x}_i \in X_i$ for $i = 1, 2$. The initial state dependent stabilization is ensured if X_i is a robust positively invariant set with respect to $g_i(y(t))$, $i = 1, 2$.*

Proof - Given a certain $y(0) \in X_i$, at the first step the applied control law is $g_i(y(t))$, and because of the invariance condition, it will be applied for all the future steps. From the ISS property, if the control law never switches, the system asymptotically converges to a neighborhood of $\overline{x}_i$ proving the claim. □

Moreover it is possible to prove that, if the "terminal positive invariant sets" containing the two equilibrium points are disjoint, the above conditions are even necessary:

Corollary: Let $\varepsilon_i = \{x : \|x - \overline{x}_i\| \leq \gamma_i(e_{\max})\}, i = 1, 2$. If $\varepsilon_1 \cap \varepsilon_2 = \emptyset$ the no-trespassing condition defined in Theorem 1 is also a necessary condition for initial state dependent stabilization if a law in the form (3) is adopted.

Proof - Let us suppose there exists a certain $y(0) \in X_i$ such that, applying (3), the system converges asymptotically to a neighborhood of $\bar{x}_i$ and, at a certain time k, $y(k) = y' \in X_j$ with $i \neq j$. Let now $y(0) = y' \in X_j$ be an initial state whose evolution converges to a neighborhood of $\bar{x}_j$. If the neighborhoods of $\bar{x}_i$ and $\bar{x}_j$ are disjointed a contradiction arises and this ends the proofs. □

B. Set Stability

In many real applications we are not interested in initial state dependent stabilization. In fact, usually, it is enough to guarantee that a control law in the form (3) would be able to make the overall closed loop system to converge to one of the possible stable equilibrium points. In particular, in this context, the condition of Theorem 1 can be relaxed as follows:

Lemma: - *Let system (1) and a piecewise continuous control law in the form (3) be given. Moreover let $g_i(y(t))$ be such that the closed-loop system $x(t+1) = f(x(t), g_i(y(t)))$ with equilibrium point $\overline{x}_i$ is ISS with restriction S_i on the initial state, with $X_i \subseteq X_i \oplus E \subseteq S_i$, and $\bar{x}_i \in X_i$ for $i = 1, 2$. The closed loop system is globally stable to the set of points $\{\bar{x}_1, \bar{x}_2\}$ if at least one of the following conditions hold true:*

- *X_1 is a robust invariant set with respect to $g_1(y(t)$*
- *X_2 is a robust invariant set with respect to $g_2(y(t))$*

Proof - Let us consider $y(t) \in X_i$. Because the applied control laws are ISS on some set $S_i \supseteq X_i$ $i = 1, 2$, if the law $g_i(y(t))$ is not robustly invariant then, either $y(t)$ converges to $\bar{x}_i$, or there exists a finite time $t' > 0$, such that $y(t') < \infty$ and $y(t') \in X_j$. In the latter case, since X_j is invariant, $y(t)$ will converge to ε_j. □

The idea behind the above lemma is that, if at least one of the two feedback laws defining (3) is attractive enough to make its definition region robustly positive invariant, then the laws cannot switch more then once. This is useful in those systems in which control actions are limited by physical constraints or actuator specifications.

Note that the above property can be emulated by the continuous-time systems by means of a *sample-and-hold* machinery. Such a solution is the idea behind the approach proposed in [13].

V. Alternative strategies

In this section, we will present an overview of strategies, able to guarantee initial state dependent or stability to a set of stable equilibrium points, that are alternative to the "abrupt" switching law (3). The common idea of the approaches we will discuss is the use of control laws in which the scheduling policy does not depend only on the instantaneous value of the output. This is made by the introduction of (sometimes implicit) memory capabilities.

A. Alternative strategies – Policy Holding

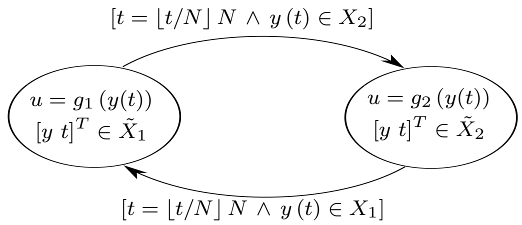

Fig. 2. The hybrid automaton which defines the policy holding control strategy. $\tilde{X}_1 = (X \times (\mathbb{R}_> - Q)) \vee (X_1 \times Q)$, $\tilde{X}_2 = (X \times (\mathbb{R}_> - Q)) \vee (X_2 \times Q)$, $Q = \{q = \lfloor t/N \rfloor N \,|\, t \in \mathbb{R}_>\}$.

Since the chattering behavior results from the changes of strategy, a possible way to address the problem is to sample and hold only the decision to switch between the control laws. Similar ideas have been explored in the stability analysis of switching systems with the introduction of the *dwelling time* (see [11]). The strategy above described can be formalized as follows

$$u(t) = \begin{cases} g_1(y(t)) & if\ y(\lfloor t/T_s \rfloor T_s) \in X_1 \\ g_2(y(t)) & if\ y(\lfloor t/T_s \rfloor T_s) \in X_2 \end{cases} \quad (7)$$

where $T_s \in \mathbb{N}$ for the discrete time case and $T_s \in \mathbb{R}_>$ for the continuous-time case. The idea of this law is that while the "dynamic control laws" are continuous-time or have their own sampling time, the switching policy is governed by a slower sampling time (T_s times slower than the dynamic one in the discrete-time case). This control scheme can be represented by means of the hybrid automaton in Figure 2.

Definition 6 (robust positive invariance at every time T) *Given system (1), the set $X_p \subset X$ is said to be "robustly positive invariant" at every time T with respect to a certain control law $u(t) = g(y(t))$ if, for all initial conditions such that $y(t_0) \in X_p$ the output $y(t_0 + kT)$ belongs to the set X_p for all $k \in \mathbb{N}$.* □

The latter definition allows us, similarly to what has been done in Section IV, to state the following result:

Theorem 2 (PH1) *Let the system (1) and the piecewise continuous control law (7) be given. Moreover let $g_i(y(t))$ be such that the closed-loop system $\delta x(t) = f(x(t), g_i(y(t)))$ with equilibrium point $\overline{x}_i$ is ISS with restriction S_i on the initial state, with $X_i \subseteq X_i \oplus E \subseteq S_i$, and $\bar{x}_i \in X_i$ for $i = 1, 2$. The initial state dependent stabilization is ensured if X_i is a robust invariant set at every time T_s.*

Proof - Given a certain $y(0) \in X_i$, the control law $g_i(y(t))$ is applied. Because ISS, if the control never switches, the system asymptotically converges to a neighborhood of $\overline{x}_i$. Because of the invariance condition, for all $t \geq T_s$, the output $y(\lfloor t/T_s \rfloor T_s) \notin X_j$ with $j \neq i$, then the control law never switches. □

If the "terminal positive invariant sets" containing the two equilibrium points are disjoint, even in this case it is possible to prove that the above conditions are also necessary.

Corollary: Given a control law in the form (7), if $\varepsilon_1 \cap \varepsilon_2 = \emptyset$ then the condition defined in Theorem 2 is also a necessary condition for initial state dependent stabilization. **Proof** - The Corollary can be proven by contradiction, following the same arguments used in the proof of the Corollary in Section IV. □

The same invariance definition allows us to write the following less conservative sufficient conditions to guarantee global stability to a set of points.

Lemma [PH2] - Let the system (1) and the piecewise continuous control law (7) be given. Moreover let $g_i(y(t))$ be such that the closed-loop system $\delta x(t) = f(x(t), g_i(y(t)))$ with equilibrium point $\overline{x}_i$ is ISS with restriction S_i on the initial state, with $X_i \subseteq X_i \oplus E \subseteq S_i$, and $\bar{x}_i \in X_i$ for $i = 1, 2$. The closed loop system is globally stable to the set of points $\{\bar{x}_1, \bar{x}_2\}$ if at least one of the following conditions hold true:

- X_1 is a robust invariant set at every time T_s with respect to $g_1(y(t))$;
- X_2 is a robust invariant set at every time T_s with respect to $g_2(y(t))$

Proof - Assume $y(t) \in X_i$. Because the applied control laws are ISS on some set S_i, with $X_i \subseteq X_i \oplus E \subseteq S_i$, $i = 1, 2$, then, if the law is not robustly invariant, either the control law never switches (and hence $y(t)$ converges to $\bar{x}_i$), or there exists a finite time $kT_s > 0$, such that $y(kT_s) < \infty$ and $y(kT_s) \in X_j$. In the latter case, the control law switches to g_j. Because X_j is invariant at every time T_s the control law will never switch again and $y(k)$ will converge to $\bar{x}_j$. □

Remark 1: Note that the control law (7) and its properties can be seen as a generalization of the abrupt (only for the discrete time case) switching control law (3) discussed in Section II. Such a control law can in fact be recovered by setting $T_s = 1$. □

To complete the analysis of the policy holding strategy, it is of interest to study its behavior in the case that an unexpected impulsive event instantaneously modifies the state values. For this reason we rewrite the system (1)

$$\begin{aligned} \delta x(t) &= f(x(t), u(t)) + w(t) \\ y(t) &= x(t) + e(t) \end{aligned} \tag{8}$$

where $w(t) \in \mathbb{R}^n = 0 \; \forall t' \neq t_0$.

Lemma [PH3] - Let us consider the system (8) and the control law (7). Then

1) if the conditions expressed in (PH2) hold, global stability to a set of points is guaranteed and the control policy switches at most 3 times;
2) if the conditions expressed in (PH1) hold, global stability to a set of points is guaranteed and the control policy switches at most once at time $t_{s1} = \left(\left\lfloor \frac{t'+T_s}{T_s} \right\rfloor T_s \right)$.

Proof - This can be proven by enumeration of the possible cases. For more details see [5]. □

B. Alternative strategies – Hysteresis

As well known in literature, a classical way to deal with many of the problems arising from the use of switching control laws, is the introduction of hysteresis regions (see for example [7]) . In order to formally define hysteresis strategies consider the finite state machine with state $q \in \{1, 2\}$, with the following initialization function $q(0) = i$ if $y(t) \in X_i$ and forced transitions

$$q(t) = 2 \mapsto q(t) = 1 \text{ if } y(t) \in X - \hat{X}_2$$

$$q(t) = 1 \mapsto q(t) = 2 \text{ if } y(t) \in X - \hat{X}_1$$

where $\hat{X}_1 \supset X_1$, $\hat{X}_2 \supset X_2$ and where $\hat{X}_1 \cap \hat{X}_2$ will be referred as the hysteresis region. By using the above finite state machine we can now introduce the following hysteresis-based control law

$$u = \begin{cases} g_1(y(t)) & if\ q(t) = 1 \\ g_2(y(t)) & if\ q(t) = 2 \end{cases} . \tag{9}$$

The idea behind this law is that, when the output $y(t)$ belongs to the hysteresis region, a switching of the control law is always forbidden with the only exception of the initial time.

To analyze the hysteresis control laws we need to introduce a new definition of positive invariance:

Definition 7 (Robust Positive Invariance with initial output in X_0)
Let the system (1) and a control law $u(t) = g(y(t))$ be given. Then, the set $X \supseteq X_0$ is said to be "robustly positively invariant with initial output in X_0" with respect to $g(y(t))$ if, for any initial output $y(0) \in X_0$, the output evolution always belong to X in forward time i.e.

$$y(t) \in X, \;\; \forall y(0) \in X_0, \;\; \forall t \geq 0 \,. \qquad \square$$

In a similar way to what has been done in the previous subsections, it is possible to state, by using the definition of robust invariance, the following results about the initial state dependent stabilization and the global stability to a set of points.

Theorem 3 *Let the system (1) and the hysteresis control law (9) be given. Moreover let $g_i(y(t))$ be such that the closed-loop system $\delta x(t) = f(x(t), g_i(y(t)))$ with equilibrium point $\overline{x}_i$ is ISS with restriction S_i on the initial state, with $X_i \subseteq X_i \oplus E \subseteq S_i$, and $\bar{x}_i \in X_i$ for $i = 1, 2$. The initial state dependent stabilization is ensured if $\hat{X}_i$ is a robustly positive invariant set with initial output in X_i.*

Proof - Because of the invariance condition, the control law never switches, then because the closed loop system is ISS $\forall y(0) \in X_i$, initial state dependent stabilization is ensured. □

Corollary - Given a law in the form (9), if $\varepsilon_1 \cap \varepsilon_2 = \emptyset$ invariance condition with initial output X_0 is also a necessary condition for initial state dependent stabilization.

Proof - Let suppose it exists a certain $y(0) \in X_i$ such that the trajectory converges asymptotically to a neighborhood of $\bar{x}_i$ and at a certain time $y(k)$ is such that $y(k) = y' \in \left(X_j - \hat{X}_i \right)$ with $i \neq j$. Let now state a certain $y(0) = y' \in X_j$ it should converge to a neighbor of $\bar{x}_j$. If the neighbor of $\bar{x}_i$ and $\bar{x}_j$ are disjointed this prove the statement by contradiction. □

Lemma - *Let the system (1) and the hysteresis control law (9) be given. Moreover let $g_i(y(t))$ be such that the closed-loop system $\delta x(t) = f(x(t), g_i(y(t)))$ with equilibrium point $\overline{x}_i$ is ISS with restriction S_i on the initial state, with $X_i \subseteq X_i \oplus E \subseteq S_i$, and $\bar{x}_i \in X_i$ for $i = 1, 2$. The closed loop system is globally stable to the set of points $\{\bar{x}_1, \bar{x}_2\}$ if at least one of the following conditions hold true:*

- *$\hat{X}_1$ is a robust invariant set with initial output in X_1 with respect to $g_1(y(t))$*
- *$\hat{X}_2$ is a robust invariant set with initial output in X_2 with respect to $g_2(y(t))$*

Proof - The Lemma can be proved following the same arguments adopted in the proof of the Lemma in Section IV. □

We complete here the analysis of the hysteresis strategy by considering the case in which an impulsive event perturbs the state of the system at a certain time t'.

Lemma - Let us consider the system (8) and the control law (9), such that $S_i \supseteq \hat{X}_i$, i=1,2. Then

1) if the conditions expressed in the previous Lemma hold, global stability to a set of points is guaranteed and the control policy switches at most 3 times;
2) if the conditions expressed in Theorem 3 hold, global stability to a set of points is guaranteed and the control policy switches at most once.

Proof - The proof follows from enumeration of the possible cases. For more details see [5]. □

Remark: Notice that the hysteresis policy is able to deal with an instantaneous change of policy for an impulsive disturbance such that $y(t) \in X_j - \hat{X}_i$. Otherwise, even if initial state stability conditions hold true, if $y(t) \in \hat{X}_i - X_i$, then a switch may occur for a certain $t > t'$. Note that, in general, this time could be arbitrarily large depending on the properties of the closed loop system, while in the case of the policy holding discussed in the previous Subsection in Lemma (PH3), it has been shown to be upper-bounded by $t' + T_s$. □

C. Alternative strategies – Mixed Strategy Holding/Hysteresis

While hysteresis has revealed capable to immediately change the control law if the state of the system is outside the "hysteresis region" but close to a desired equilibrium point, only holding strategies can guarantee a fixed upper-bound to the maximum time before a switch. This fact suggests to look for "mixed" strategies in order to take advantage of both the above properties at the same time.

We then consider the following mixed holding / hysteresis control law

$$u = \begin{cases} g_1(y(t)) & if\ y(\lfloor t/T_s \rfloor T_s) \in X_1 \vee y(t) \in \overline{X}_1 \\ g_2(y(t)) & if\ y(\lfloor t/T_s \rfloor T_s) \in X_2 \vee y(t) \in \overline{X}_2 \end{cases} \quad (10)$$

where $\overline{X}_1 \subseteq X_1, \overline{X}_2 \subseteq X_2$ and T_s are the design parameter.

By construction we assume that the sets $X - \overline{X}_i$, $i = 1, 2$, are robustly positive invariant with initial output in X_j, for $i \neq j$. Under this hypothesis, the properties of system (1) equipped with the law (10) are determined only by the robust positive invariance at every time T_s of the sets X_i, according to the results given in Theorem 2 and Lemma I for the policy holding strategy. The case of an impulsive disturbance is then of interest and it is summarized by the following Lemma.

Lemma - Consider the system (8) and the control law (10). Let the set $\overline{X}_i$ be such that $X - \overline{X}_i$ is robustly positive invariant with initial output in X_j, for $i \neq j$, $i = 1, 2$. Then

1) if the conditions expressed in (PH2) hold, global stability to a set of points is guaranteed and the control policy switches at most 3 times;
2) if the conditions expressed in (PH1) hold, global stability to a set of points is guaranteed and the control policy switches at most once at time $t_s \leq t_{s1} = \left(\left\lfloor \frac{t'+T_s}{T_s} \right\rfloor T_s \right)$.

Proof - This can be proven by enumeration of the possible cases. For more details see [5]. □

VI. Conclusion

In this paper the problem of chattering due to measurement noise for systems controlled by a piecewise continuous control law has been studied. Conditions that ensure global stabilization have been also studied. Moreover further alternative strategies, employable both for continuous and discrete-time systems have been proposed and deeply investigated. A particular attention has been given to "holding strategies" and "hysteresis policies" and their relative advantages and disadvantages remarked. Finally a hybrid policy combining the advantages of holding and hysteresis ideas has been briefly proposed.

References

[1] A. Bemporad, G. Ferrari-Trecate, and M. Morari. Observability and controllability of piecewise affine and hybrid systems. *IEEE Transactions on Automatic Control*, 45(10):1864–1876, 2000.

[2] S.P. Bhat and D.S. Bernstein. A topological obstruction to global asymptotic stabilization of rotational motion and the unwinding phenomenon. *Proceedings of the American Control Conference*, 1998.

[3] F. Blanchini. Set invariance in control – a survey. *Automatica*, 35:1747–1768, 2002.

[4] C. Cai and A.R. Teel. Input-output-to-state stability for discrete-time systems. *Automatica*, 2008.

[5] E. Garone, R. Naldi, E. Frazzoli, and A. Casavola. Notes on discontinuos control systems in the presence of measurment noise. *Internal Report*, 2008.

[6] R. Goebel, J. Hespanha, A.R. Teel, C. Cai, and R.G. Sanfelice. Hybrid systems: generalized solutions and robust stability. In *Proceedings of NOLCOS 2004*, 2004.

[7] R. Hudon, J.G. Paquet, and J.C. Gille. On the hysteresis of on-off elements. *IEEE Transactions on Automatic Control*, 14:52–55, 1969.

[8] J. Imura. Well-posedness analysis of switch-driven piecewise affine systems. *IEEE Transactions on Automatic Control*, 48(11):1926–1935, 2003.

[9] Z.P. Jiang, E.D. Sontag, and Y. Wang. Input-to-state stability for discrete-time nonlinear systems. *Proceedings of 14th IFAC World Congress*, 1999.

[10] C.M. Kellet, H. Shim, and A.R. Teel. Robustness of discontinuous feedback via sample and hold control. *Proceedings of American Control Conference 2002*, 2002.

[11] D. Liberzon. Switching in systems and control. *Birkhauser*, 2003.

[12] R.G. Sanfelice, M.J. Messina, S.E. Tuna, and A.R. Teel. Robust hybrid controllers for continuous-time systems with applications to obstacle avoidance and regulation to disconnected set of points. *Proceedings of American Control Conference 2006*, 2006.

[13] E.D. Sontag. Clocks and insensitivity to small measurement errors. *Control Optimisation and Calculus of Variations*, 1999.

[14] E.D. Sontag and Y. Wang. New characterizations of input-to-state stability. *IEEE Transactions on Automatic Control*, 41:1283–1294, 1996.

[15] A. Teel. A nonlinear small gain theorem for the analysis of control systems with saturations. *IEEE Transactions on Automatic Control*, 41:1256–1270, 1996.

[16] J.T. Wen and K.K. Delgado. The attitude control problem. *IEEE Transactions on Automatic Control*, 36:1148–1162, 1991.