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LQG Control Over Lossy TCP-like Networks With Probabilistic Packet Acknowledgements

E. Garone, B. Sinopoli and A. Casavola

Abstract— **This paper focuses on control applications over lossy data networks. Sensor data is transmitted to an estimation-control unit over a network and control commands are issued to subsystems over the same network. Sensor, control and acknowledgement packets may be randomly lost according to a Bernoulli process. In this context, the discrete-time Linear Quadratic Gaussian (LQG) optimal control problem is considered. We can show how the partial loss of acknowledgements makes the optimal control law a nonlinear function of the information set. For the special case of complete state observation we can compute the optimal controller and show that the stability range increases monotonically with the arrival rate of the acknowledgement packets.**

I. INTRODUCTION

This paper is concerned with the design and analysis of control systems where components are connected via packet-based communication networks. This requires a generalization of classical control techniques to explicitly take into account the stochastic nature of the communication channel. In recent years these kinds of problems have drawn considerable attention in the academic world, focusing on estimation ([1],[2] and [3],[4],[5] and [6]) and optimal control problems (see [7], [8] and [9]). In particular the problem considered in this paper is a generalized formulation of the Linear Quadratic Gaussian (LQG) optimal control problem where the arrival of both measurement and control packets are modeled as random processes, whose parameters are related to the characteristics of the communication channel. Accordingly, independent Bernoulli processes are considered, with parameters $\overline{\gamma}$ and $\overline{\nu}$ governing the packet losses between the sensors and the estimation-control unit and between the latter and the actuation points. The key issue relating to properly design networked control systems is to clearly understand which information is available at each time instant to the controller. It is usual to distinguish between TCP-like protocols, where packet acknowledgements are guaranteed at each time instant, and UDP-like protocols in the case that no acknowledgement mechanism is provided [10]. In many real cases this distinction is too simplistic because it is impossible to guarantee a perfectly deterministic acknowledgement through an unreliable channel. In this paper we will deal with the control design problem for networked system in which acknowledgement packets can be lost according to a Bernoulli process of parameter $\overline{\theta}$.

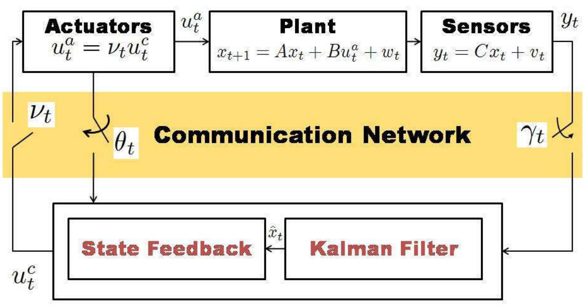

Fig. 1. **Overview of the system.** Architecture of the closed loop system over a communication network. The binary random variables ν_t, γ_t and θ_t indicates whether packets are transmitted successfully.

Previous work ([1], [7], [11]) has shown the existence of a critical domain of values for the parameters of the Bernoulli arrival processes, $\overline{\nu}$ and $\overline{\gamma}$, outside of which a transition to instability occurs and the optimal controller fails to stabilize the system. In particular, it has been shown that the classical separation principle holds in the TCP-like protocols, the optimal control is linear and the critical arrival probabilities for the control and observation channels are independent each other. On the contrary, under UDP-like protocols, no separation principle arises, the optimal control is in general nonlinear and, in the case of complete observability, critical arrival probabilities are coupled and define more restrictive stability regions, as shown in Figure 2.

In this paper we will show that in the "quasi-TCP like" context considered here, even if the same "structural problems" of the UDP-like protocols arise (nonlinearity of the optimal law and no separation principle), in the case of complete observability, the stability range of the system increases with the arrival rates of the acknowledgement packets. Furthermore, we can show how such a stability range converges to that achievable under the TCP-like protocol, as the probability of acknowledgement packet drops tends to zero.

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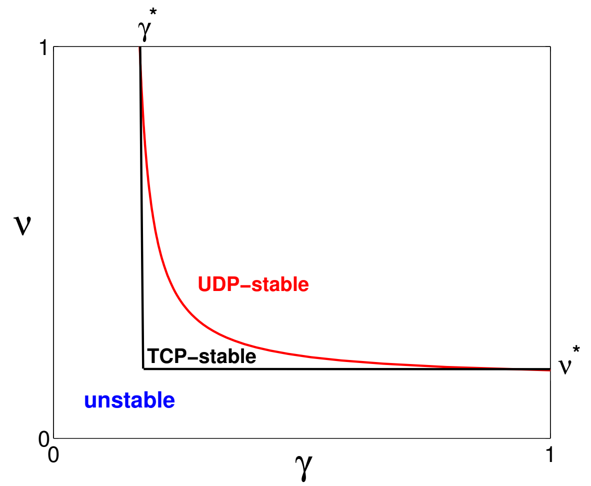

Fig. 2. Region of stability for UDP-like and TCP-like optimal control relative to measurement packet arrival probability $\bar{\gamma}$, and the control packet arrival probability $\bar{\nu}$ in the case $C = I$.

The remainder of this paper is organized as follows. Section 2 provides the problem formulation. In Section 3 the Single Channel problem is studied: nonlinearity of the optimal control is pointed out and the optimal control for the complete observability case considered. Section 4 generalizes the previous Section's results to the multi-channel case. Section 5 provides an example, Finally, in Section 6, conclusions are provided.

II. Problem Formulation

Consider the following linear stochastic system with intermittent observation and control packets:

$$\begin{array}{rcl} x_{k+1} & = & Ax_k + Bu_k^a + \omega_k, \\ u_k^a & = & \mathrm{N}_k u_k + [I_{m\times m} - \mathrm{N}_k]\, u_k^l, \\ y(k) & = & \Gamma_k (Cx_k + v_k), \end{array} \tag{1}$$

where $x_k \in \mathcal{R}^n$ is the state vector, $y_k \in \mathcal{R}^p$ is the output vector, $(x_0 \in \mathcal{R}^n, w_k \in \mathcal{R}^n, v_k \in \mathcal{R}^p)$ are Gaussian, uncorrelated, white, with mean $(\overline{x_0}, 0, 0)$ and covariance (P_0, Q, R) respectively. Moreover

$$N_k = \begin{bmatrix} \nu_{1,k} & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & \nu_{m,k} \end{bmatrix}, \tag{2}$$

$$\Gamma_k = \begin{bmatrix} \gamma_{1,k} & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & \gamma_{p,k} \end{bmatrix}, \tag{3}$$

where $(\gamma_{i,k})$, $i = 1,\dots,p$ and $(\nu_{j,k})$ $j = 1,\dots,m, \forall k \in \mathbb{Z}$, are binary variables modeling the successful transmission of the information from the i-th sensor and to the j-th actuator at time k. $u_k^a \in \mathcal{R}^m$ is the effective control input applied to the actuators while $u_k \in \mathcal{R}^m$ denotes the desired control input computed by the controller. Finally $u_k^l \in \mathcal{R}^m$ is the signal locally provided to the actuators in the case $N_k = 0_{m\times m}$ (all packets to the actuators are lost). While it is possible to choose $u^l(k)$ in several ways, the most common strategies are the following:

1) zero-input scheme: $u_k^l = 0$
2) hold-input scheme: $u_k^l = u_{k-1}^a$

Here we will deal with the zero-input scheme.

Because groups of sensors/actuators could send/receive their data in the same packet, we will suppose that the information transmission is organized in independent sensor and actuator clusters . This means we can rewrite Γ_k and N_k as follows:

$$\Gamma_k = I_{p\times p} - \prod_{i=1}^{p'} \left(I_{p\times p} - \gamma'_{i,k} diag\{g_i\}\right) \tag{4}$$

$$N_k = I_{m\times m} - \prod_{i=1}^{m'} \left(I_{m\times m} - \nu'_{i,k} diag\{\eta_i\}\right) \tag{5}$$

where $\gamma'_{i,k}$ and $\nu'_{j,k}$ are i.d.d. Bernoulli processes with probabilities of successful transmission $\overline{\gamma'_i} = P\left(\gamma'_{i,k} = 1\right), i = 1,\dots,p'$ and $\overline{\nu'_j} = P\left(\nu'_{j,k} = 1\right), i = 1,\dots,m'$. $g_i, i = 1,\dots,m'$ and $\eta_i, i = 1,\dots,p'$ are vectors of length p and m respectively such that:

- $(g_i)_j = 1$ $((\eta_i)_j = 1)$ if the j-th sensor (actuator) belongs to the i-th cluster
- $(g_i)_j = 0$ $((\eta_i)_j = 0)$ if the j-th sensor (actuator) does not belong to the i-th cluster

A key point toward the design of any control strategy is the definition of the *Information Set* available to the controller at each time instant. It is usual in literature (see [10]) to refer to the following two information sets

$$I_k = \begin{cases} F_k = \{\Gamma_t y_t, \Gamma_t, \mathrm{N}_{t-1} | t = 0,..,k\} & TCP-like \\ G_k = \{\Gamma_t y_t, \Gamma_t | t = 0,..,k\} & UDP-like \end{cases} \tag{6}$$

The difference between the two Information Sets is the acknowledgement of the actually arrived packets to the actuators i.e. the matrix N_{k-1}.

While the "TCP-like" case has several desirable properties (separation principles, linear quadratic gaussian optimal control, etc...), it is well known that the availability of deterministic "perfect" acknowledgements, in the case the acknowledgement packets use unreliable channels, is theoretically impossible since it represents a particular case of the *two-armies problem* (see [12]). On the other hand if "UDP-like" protocols [7] are employed (see Figure 2) performance and stability regions are highly affected, due to the fact that no "real" information on the actual input is exploited. In many practical cases, it is reasonable to use communication channels where acknowledgements are provided although they can be dropped, i.e. we have a non-zero probability of losing the acknowledgment packet from the channel j. To formalize this assumption we introduce the matrix

$$\Theta_k = \begin{bmatrix} \theta_{1,k} & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & \theta_{m,k} \end{bmatrix}, \tag{7}$$

where $\theta_{i,k}$ is the acknowledgement event from the i-th actuator at time k. It can be rewritten as

$$\Theta_k = I_{m\times m} - \prod_{i=1}^{m'} \left(I_{m\times m} - \theta'_{i,k} diag\left\{\eta_i\right\}\right) \quad (8)$$

where $\theta'_{i,k}$ are i.i.d. Bernoulli processes with $\overline{\theta'_i} = P\left(\theta'_{i,k}=1\right), i = 1,\ldots,m'$. The information structure of a networked system with stochastic acknowledgments is the following:

$$E_k = \{\Gamma_k y_k, \Gamma_k, \Theta_{k-1}, \Theta_{k-1}N_{k-1}|k=0,..,t\}. \quad (9)$$

Let us now define $u^{N-1} = \{u_0, u_1, \ldots, u_{N-1}\}$ as the set of all the input values between 0 and $N-1$. In this paper we will analyze the LQG control problem, i.e. we will look for a control input sequence u^{N-1*}, function of the information set E_k, that solves the following optimization problem:

$$J_N^*(\bar{x}_0, P_0) = min_{u_k = g_k(E_k)} J_N(u^{N-1}, \bar{x}_0, P_0), \quad (10)$$

where the cost function $J_N\left(u^{N-1}, \overline{x}_0, P_0\right)$ is defined as follows:

$$\begin{array}{l} J_N\left(u^{N-1}, \overline{x}_0, P_0\right) = \\ = E\left[x_N^T W_N x_N + \sum\limits_{k=0}^{N-1} x_k^T W_k x_k + u_k^{aT} U_k u_k^a \middle| u^{N-1}, \overline{x}_0, P_0\right] \end{array}. \quad (11)$$

Because the general multichannel formulation requires a large use of notation that would affect negatively the intuitive nature of the results, we will first concentrate the treatment on the single channel case, i.e. $m' = 1$ and $n' = 1$ and provide for this case the main results. After that, we will present the results for the multi-channel case with fewer details. Due to lack of space, many of the proofs will be omitted and the interested reader is referred to [13] for details.

III. SINGLE INPUT/OUTPUT CHANNEL CASE

A. Estimator design

If $m' = 1$ and $n' = 1$, system (1) becomes

$$\begin{array}{l} x_{k+1} = Ax_k + \nu_k B u_k + \omega_k \\ y_k = \gamma_k C x_k + v_k \end{array} \quad (12)$$

and $\Theta_k = \theta_k$. By the knowledge of the information set (9), the one-step prediction can be written as:

$$\hat{x}_{k+1|k} = A\hat{x}_{k|k} + \theta_k \nu_k B u_k + (1-\theta_k)\overline{\nu} B u_k. \quad (13)$$

Using (13) it is possible to rewrite the predicted error as follows:

$$\begin{array}{l} e_{k+1|k} = x_{k+1} - \hat{x}_{k+1|k} = Ax_k + \nu_k B u_k + \omega_k - A\hat{x}_{k|k} \\ +\theta_k \nu_k B u_k - (1-\theta_k)\overline{\nu} B u_k = \\ = Ae_{k|k} + (\nu_k - \theta_k\nu_k - (1-\theta_k)\overline{\nu}) B u_k + \omega_k \end{array}. \quad (14)$$

We can then compute the associated error covariance one-step prediction:

$$\begin{array}{l} P_{k+1|k} = E\left[e_{k+1|k} e_{k+1|k}^T | E_k, \theta_k, \theta_k\nu_k\right] = \\ = E\left[Ae_{k|k}e_{k|k}^T A | E_k\right] + E\left[\omega_k\omega_k^T | E_k\right] + \\ +E\left[(\nu_k - \theta_k\nu_k - (1-\theta_k)\overline{\nu})^2 | E_k, \theta_k, \theta_k\nu_k\right] B u_k u_k^T B^T, \end{array}$$

obtaining

$$P_{k+1|k} = AP_{k|k}A^T + Q + (1-\theta_k)(1-\overline{\nu})\overline{\nu}\left[Bu_k u_k^T B^T\right]. \quad (15)$$

Equations (13), (14) and (15) represent the predictions of the Kalman Filter for the system (12). The correction steps, instead, are the classical ones considered in [14]:

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + \gamma_{k+1}K_{k+1}\left(y_{k+1} - Cx_{k+1|k}\right) \quad (16)$$

$$P_{k+1|k+1} = P_{k+1|k} - \gamma_{k+1}K_{k+1}CP_{k+1|k} \quad (17)$$

$$K_{k+1} = P_{k+1|k}C^T\left(CP_{k+1|k}C^T + R\right)^{-1} \quad (18)$$

Remark 1: Note that:

$$\begin{array}{ll} \theta_k = 1 \Rightarrow & P_{k+1|k} = AP_{k|k}A + Q \\ \theta_k = 0 \Rightarrow & P_{k+1|k} = AP_{k|k}A + Q + \overline{\nu}(1-\nu)\left[Bu_k u_k^T B^T\right]. \end{array}$$

This implies that, at each time k, the prediction switches between the "TCP-like" predictions or the "UDP-like" one, depending on the instantaneous value of θ_k.

B. Optimal Control - general case

Here we will show that, in the presence of stochastic acknowledgements, the optimal control law is not a linear function of the state estimate and that the estimation and control design cannot be treated separately. In order to prove such a statement, it is sufficient to consider the following simple counterexample. Consider a simple scalar discrete-time Linear Time-Invariant (LTI) System with a single actuator and a single sensor, i.e. A=B=C=W$_N$=W$_k$=R=1,U$_k$=Q=0. We can define the value function

$$V(N) = E\left[x_N^T W_N x_N | E_N\right] = E\left[x_N^2 | E_N\right];$$

for $k = N-1$ we will have:

$$\begin{array}{l} V_{N-1}(x_{N-1}) = \min\limits_{u_N} E\left[x_{N-1}^2 + V_N(x_N) | E_{N-1}\right] = \\ = \min\limits_{u_N} E\left[x_{N-1}^2 + x_N^2 | E_{N-1}\right] = \\ = \min\limits_{u_N} E\left[x_{N-1}^2 + (x_{N-1} + \nu_{N-1}u_{N-1})^2 | E_{N-1}\right] \end{array} \quad (19)$$

and then finally

$$\begin{array}{l} V_{N-1}(x_{N-1}) = E\left[2x_{N-1|N-1}^2 | E_{N-1}\right] + \\ + \min\limits_{u_N} \overline{\nu}\left(u_{N-1}^2 + 2\hat{x}_{N-1|N-1}u_{N-1}\right) \end{array} \quad (20)$$

If we differentiate the latter, we obtain the following optimal input :

$$u_{N-1}^* = -\hat{x}_{N-1|N-\hat{1}} \quad (21)$$

If we substitute (21) in (19) the cost becomes:

$$\begin{array}{l} V_{N-1}(x) = E\left[2x_{N-1}^2 | E_{N-1}\right] - \overline{\nu}\,\hat{x}_{N-1|N-1}^2 = \\ = (2-\overline{\nu})E\left[x_{N-1}^2 | G\right] - \overline{\nu}P_{N-1|N-1}. \end{array} \quad (22)$$

Let us focus now on the covariance matrix:

$$\begin{array}{l} P_{N-1|N-1} = P_{N-1|N-2} - \gamma_{N-1}\frac{P_{N-1|N-2}^2}{(P_{N-1|N-2}+1)} = \\ = P_{N-1|N-2} - \gamma_{N-1}\left(P_{N-1|N-2} - 1 + \frac{1}{(P_{N-1|N-2}+1)}\right) \end{array} \quad (23)$$

because of

$$P_{N-1|N-2} = P_{N-2|N-2} + (1-\theta_{N-2})(1-\overline{\nu})\overline{\nu}u_{N-2}^2 \quad (24)$$

then:

$$\begin{aligned}&E\left[P_{N-1|N-1}|E_{N-2}\right] = P_{N-2|N-2} + \left(1-\overline{\theta}\right)(1-\overline{\nu})\overline{\nu}u_{N-2}^2 + \\ &-\overline{\gamma}\left(P_{N-2|N-2} + \left(1-\overline{\theta}\right)(1-\overline{\nu})\overline{\nu}\,u_{N-2}^2 + \overline{\theta}\frac{1}{P_{N-2|N-2}}\right. \\ &\left.-1 + \left(1-\overline{\theta}\right)\frac{1}{P_{N-2|N-2}+(1-\overline{\nu})\overline{\nu}u_{N-2}^2}\right).\end{aligned} \quad (25)$$

Finally we get

$$\begin{aligned}&V_{N-2}(x) = \min_{u_{N-2}} E\left[x_{N-2}^2 + V_{N-1}(x_{N-1})|E_{N-2}\right] = \\ &= (3-\overline{\nu})\,E\left[x_{N-1}^2|E_{N-2}\right] + \min_{u_{N-2}} P_{N-2|N-2} + \\ &+ \left(1-\overline{\theta}\right)(1-\overline{\nu})\overline{\nu}u_{N-2}^2 + \\ &-\overline{\gamma}\left(P_{N-2|N-2} + \left(1-\overline{\theta}\right)(1-\overline{\nu})\overline{\nu}u_{N-2}^2 - 1 + \right. \\ &\left. + \overline{\theta}\frac{1}{P_{N-2|N-2}} + \left(1-\overline{\theta}\right)\frac{1}{P_{N-2|N-2}+(1-\overline{\nu})\overline{\nu}u_{N-2}^2}\right)\end{aligned} \quad (26)$$

The first terms within the last parenthesis in (26) are convex quadratic functions of the control input u_{N-2}; however, the last term is not such. Therefore, the optimal control law is, in general, a nonlinear function of the information set E_k. By inspection we can state the following result

Theorem 1: Let us consider the stochastic system defined in Equation (12) with horizon $N \geq 2$. Then:

- if $\overline{\theta} < 1$ (TCP-like case), the separation principle does not hold
- The optimal control feedback $u_k = g_k^*(E_k)$ that minimizes the cost functional defined in Equation (11) is, in general, a nonlinear function of information set E_k
- The optimal control feedback $u_k = g_k^*(E_k)$ is a linear function of the estimated state if and only if one of the following conditions hold true:
  - $\overline{\theta} = 1$
  - $Rank(C) = n$ and $R = 0$

□

In the next subsection we will focus on the case where $Rank(C) = n$, and $R = 0$. In particular we will compute the optimal control, and we will show that, in the infinite horizon scenario, the optimal state-feedback gain is constant, i.e. $L_k^* = L^*$ and can be computed as the solution of a convex optimization problem.

C. Optimal Control – Rank(C)=n, R=0 case

Without loss of generality we can assume $C = I$. Because of the hypothesis of no measurement noise, i.e. $R = 0$, it is possible to simply measure the state x_k when a packet is delivered. The estimator equations then simplify in the following way:

$$K_{k+1} = I \quad (27)$$

$$\begin{aligned}P_{k+1|k} &= AP_{k|k}A + Q + \\ &+ (1-\theta_k)(1-\overline{\nu})\overline{\nu}\left[Bu_ku_k^TB^T\right]\end{aligned} \quad (28)$$

$$\begin{aligned}&P_{k+1|k+1} = (1-\gamma_{k+1})P_{k+1|k} = (1-\gamma_{k+1}) \\ &\left(AP_{k|k}A + Q + (1-\theta_k)(1-\overline{\nu})\overline{\nu}\left[Bu_ku_k^TB^T\right]\right)\end{aligned} \quad (29)$$

$$\begin{aligned}&E\left[P_{k+1|k+1}|E_k\right] = \\ &= (1-\overline{\gamma})\left(AP_{k|k}A + Q + \left(1-\overline{\theta}\right)(1-\overline{\nu})\overline{\nu}\left[Bu_ku_k^TB^T\right]\right).\end{aligned} \quad (30)$$

In the last equation the independence of $E_k, \gamma_{k+1}, \theta_k$ is exploited. Following the classical dynamic programming approach to the optimal control, we assume that the value function $V_k^*(x_k)$ can be written as follows:

$$\begin{aligned}&V_k(x_k) = \hat{x}_{k|k}^TS_k\hat{x}_{k|k} + trace\left(T_kP_{k|k}\right) + trace\left(D_kQ\right) = \\ &= E\left[x_{k|k}^TS_kx_{k|k}\right] + trace\left(H_kP_{k|k}\right) + trace\left(D_kQ\right)\end{aligned} \quad (31)$$

for each $k = N, \ldots, 0$ where $H_k = T_k - S_k$. This is clearly true for $k = N$; in fact, we have:

$$V_N(x_N) = E\left[x_N^TW_Nx_N|E_N\right] = \hat{x}_{N|N}^TW_N\hat{x}_{N|N} + trace\left(W_NP_{N|N}\right)$$

Therefore the statement is satisfied by $S_N = T_N = W_N, D_N = 0$. Let us suppose that Equation (31) is true for $k+1$ and we show by induction that it holds true for k:

$$\begin{aligned}&V_k(x_k) = \min_{u_k} E\left[x_k^TW_kx_k + \nu_ku_k^TU_ku_k + V_{k+1}(x_{k+1})|E_k\right] = \\ &\min_{u_k} E\left[x_k^TW_kx_k|E_k\right] + \overline{\nu}u_k^TU_ku_k + E\left[x_{k+1}^TS_{k+1}x_{k+1}|E_k\right] + \\ &+ trace\left(H_{k+1}P_{k+1|k+1}\right) + trace\left(D_{k+1}Q\right) = \\ &= \min_{u_k} E\left[x_k^TW_kx_k|E_k\right] + \overline{\nu}_ku_k^TU_ku_k + trace \\ &\left[H_{k+1}\left((1-\overline{\gamma})\left(AP_{k|k}A + Q + (1-\theta_k)\overline{\nu}\left(1-\overline{\nu}\right)\left[Bu_ku_k^TB^T\right]\right)\right)\right] \\ &+ E\left[\left(Ax_{k|k} + \theta_k\nu_kBu_k + (1-\theta_k)\overline{\nu}Bu_k\right)^TS_{k+1}\right. \\ &\left.\left(Ax_{k|k} + \theta_k\nu_kBu_k + (1-\theta_k)\overline{\nu}Bu_k\right)|E_k\right] + trace\left(D_{k+1}Q\right).\end{aligned}$$

Exploiting the convexity of $V_k(x_k)$ w.r.t. u_k and by further manipulation, we can find its minimizer that is the solution of $\partial V_k(x_k)/\partial u_k = 0$:

$$u_k^* = -\left(U_k + B^T(S_{k+1} + \bar{\alpha}H_{k+1})B\right)^{-1}\left(B^TS_{k+1}Ax_{k|k}\right) = L_kx_{k|k}, \quad (32)$$

where $\overline{\alpha} = (1-\overline{\gamma})\left(1-\overline{\theta}\right)(1-\overline{\nu})\overline{\nu}$. The optimal control is a linear function of the estimated state $x_{k|k}$. Substituting back (32) into the value function and by proceeding with further manipulations we get:

$$\begin{aligned}&V_k(x_k) = trace\left(\left(D_{k+1} + (1-\overline{\gamma})H_{k+1}\right)Q\right) + \\ &+ E\left[x_{k|k}^T\left(W_k + A^TS_{k+1}A\right)x_{k|k} + \left(\overline{\nu}\left(x_{k|k}^TA^TS_{k+1}B\right)L_kx_{k|k}\right)\right] + \\ &+ trace\left(\left(\left((1-\overline{\gamma})A^TH_{k+1}A - \overline{\nu}A^TS_{k+1}BL_k\right)P_{k|k}\right)\right).\end{aligned} \quad (33)$$

From the last equation we see that the value function can be written as in Equation (31) if and only if the following equations are satisfied:

$$S_k = W_k + A^TS_{k+1}A. + \overline{\nu}\left(A^TS_{k+1}B\right)L_k \quad (34)$$

$$T_k = (1-\overline{\gamma})A^TT_{k+1}A + W_k + \overline{\gamma}A^TS_{k+1}A \quad (35)$$

$$D_k = D_{k+1} + (1-\overline{\gamma})T_{k+1} + \bar{\gamma}S_{k+1} \quad (36)$$

Remark 2: Notice that, if $\overline{\theta} \rightarrow 0$, the control design system soon regresses to the UDP-like case studied in [1] The optimal minimal cost for the finite horizon, $J_N^* = V_0(x_0)$ is then given by:

$$J_N^* = \bar{x_0}^TS_0x_0 + trace(S_0P_0) + trace(D_kQ).$$

For the infinite horizon optimal controller, necessary and sufficient conditions for the average minimal cost

$J_\infty^* = \lim_{N\to\infty} J_N^*$ to be finite are that the coupled iterative Equations (35) and (34) should converge to a finite value S_∞ and T_∞ as $N \to \infty$.

Theorem 2: Consider the system (12) and consider the problem of minimizing the cost function (11) within the class of admissible policies $u_k = f(E_k)$. Assume also that $R = 0$ and C is square and invertible. Then:

1) The optimal estimator gain is constant and in particular $K_k = I$ if $C = I$.
2) The infinite horizon optimal control exists if and only if there exist positive definite matrices $S_\infty, T_\infty > 0$ such that $S_\infty = \Phi_S(S_\infty, T_\infty)$ and $T_\infty = \Phi_T(S_\infty, T_\infty)$, where Φ_S and Φ_T are given by

$$\Phi_S(S_k, W_k) = W_k + A^T S_k A - \overline{\nu}\left(A^T S_k B\right) \left(U_k + B^T\left((1-\overline{\alpha})S_{k+1} + \overline{\alpha}T_{k+1}\right)B\right)^{-1}\left(B^T S_{k+1} A\right) \quad (37)$$

$$\Phi_T(S_k, T_k) = (1-\overline{\gamma}) A^T T_{k+1} A + W_k + \overline{\gamma} A^T S_{k+1} A \quad (38)$$

3) The infinite horizon optimal controller gain is constant: $\lim_{k\to\infty} L_k = L_\infty$

$$L_\infty = -\left(U + B^T\left((1-\overline{\alpha})S_\infty + \overline{\alpha}T_\infty\right)B\right)^{-1}\left(B^T S_\infty A\right) \quad (39)$$

4) A necessary condition for the existence of $S_\infty, T_\infty > 0$ is:

$$1 - |A|^2\left(1 - \frac{\overline{\nu}}{(1-\overline{\alpha}) + \overline{\alpha}\frac{\gamma|A|2}{1-(1-\gamma)|A|^2}}\right) \geq 0, \qquad \overline{\gamma} > 1 - \frac{1}{|A|^2} \quad (40)$$

where $|A| = \max_i |\lambda_i(A)|$ is the largest eigenvalue of the matrix A. This condition is also sufficient if B is square and invertible.
5) The expected minimum cost for the infinite horizon scenario converges to:

$$J_\infty^* = \lim_{k\to\infty} \frac{1}{N} J_N^* = trace\left(\left((1-\overline{\gamma})T_k + \overline{\gamma}S_k\right)Q\right) \quad (41)$$

Proof: see [13] ■

IV. Generalization to the Multichannel Case

Here, following the same reasoning of the previous Section, we will generalize the obtained results to the multi-channel case. Due to space constraints, here we will just summarize the principal differences with the single channel case. For details please refer to [13].

A. Optimal Observer

The prediction step of the Kalman filter shown in equation (13), (14) and (15) for the single input case becomes

$$\hat{x}_{k+1|k} = A\hat{x}_{k|k} + B\Theta_k N_k u_k + B(1-\Theta_k)\bar{N}u_k \quad (42)$$

$$e_{k+1|k} = Ae_{k|k} + B(I-\Theta_k)\left(N_k - \bar{N}\right)u_k + \omega_k \quad (43)$$

$$P_{k+1|k} = AP_{k|k}A^T + Q + B(1-\Theta_k)\left(\Psi(u_k, \bar{N})\right)(1-\Theta_k)B^T \quad (44)$$

where the following shorthand is introduced

$$\bar{N} = E[N_k] = I_{m\times m} - \prod_{i=1}^{m'}\left(I_{m\times m} - \bar{\nu}_i' diag\{\eta_i\}\right), \quad (45)$$

$$N_I = I_{m\times m} - \prod_{i\in I\cup\{0\}}\left(I_{m\times m} - diag\{\eta_i\}\right), \quad (46)$$

$$\Psi(u_k, \bar{N}) = \sum_{I\in 2^{\Im}}\left(\left(\prod_{i\in I}\bar{\nu}_i' \prod_{i\notin I}(1-\bar{\nu}_i')\right)\left[\left(N_I - \bar{N}\right)u_k u_k^T\left(N_I - \bar{N}\right)\right]\right). \quad (47)$$

where $I \subseteq \Im \equiv \{1, ..., m'\}$ is a set of index and $\eta_0 = 0_m$. The correction steps, instead, remain the ones shown in [15]:

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1}\Gamma_{k+1}^m\left(y_{k+1} - Cx_{k+1|k}\right) \quad (48)$$

$$P_{k+1|k+1} = P_{k+1|k} - K_{k+1}\Gamma_{k+1}^m C P_{k+1|k} \quad (49)$$

$$K_{k+1} = P_{k+1|k}C^T\Gamma_{k+1}^{mT}\left(\Gamma_{k+1}^m C P_{k+1|k} C^T \Gamma_{k+1}^{mT} + \Gamma_{k+1}^m R \Gamma_{k+1}^{mT}\right)^{-1} \quad (50)$$

where Γ_k^m is the matrix of the nonzero rows of Γ_k.

B. Optimal Control

In this subsection we generalize the above theorem for the multi-channel case. In the general case, the following result can be stated:

Theorem 3: Let us consider the stochastic system defined in (12) with horizon $N \geq 2$. Then:

- if $\exists i : \theta_i < 1$, the separation principle does not hold
- The optimal control feedback $u_k = g_k^*(E_k)$ that minimizes the cost functional defined in Equation (11) is, in general, a nonlinear function of information set E_k
- The optimal control feedback $u_k = g_k^*(E_k)$ is a linear function of the estimated state if and only if one of the following conditions hold true:
  - $\overline{\theta}_i = 1, \forall i$
  - $Rank(diag\{g_i\}C) = n, i = 1, \ldots, p'$ and $R = 0$

□

It is worth noticing that the conditions $Rank(diag\{g_i\}C) = n$ and $R = 0$ are equivalent to the case whereby every sensor data packet contains the actual value of the whole state. System (1) is then equivalent to

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k^a + \omega_k, \\ u_k^a &= \mathrm{N}_k u_k + \left[I_{m\times m} - \mathrm{N}_k\right] u_k^l, \\ y(k) &= \gamma_k x_k, \end{aligned} \quad (51)$$

where

$$\gamma_k = 1 - \prod_{i=1}^{p'}\left(1 - \gamma_{i,k}'\right).$$

This means that the optimal control is linear only when the sensing apparatus is able to perfectly measure and deliver the full state. For such a case it is possible to extend the results previously derived in the following manner:

Theorem 4: Consider the system (1) and consider the problem of minimizing the cost function (11) within the class of admissible policies $u_k = f(E_k)$. Assume also that $R = 0$ and C is square and invertible. Then:

(a) The optimal estimator gain is constant and in particular $K_k = I$ if $C = I$.

(b) The optimal control is linear and is

$$u_k^* = -\left[\Omega_{\bar{N}\bar{\Theta}}\left(S_{k+1}, H_k\right)\right]^{-1}\left(\overline{N}\right)B^T S_{k+1} A x_{k|k} = L_k x_{k|k} \tag{52}$$

where

$$\Omega_{\bar{N}\bar{\Theta}}\left(S_{k+1}, H_k\right) =$$
$$= \sum_{\substack{I \in 2^{\Im} \\ I_\theta \in 2^{\Im}}} \left[\left(\prod_{\substack{i \in I \\ j \in I_\Theta}} \bar{\nu}_i' \bar{\theta}_j' \prod_{\substack{i \notin I \\ j \notin I_\Theta}} \left(1 - \bar{\theta}_j'\right)\left(1 - \bar{\nu}_i'\right)\right)\right.$$
$$\left(N_I U_k N_I + N_I \Theta_{I_\Theta} B^T S_{k+1} B \Theta_{I_\Theta} N_I + \right.$$
$$+ \left(N_I - \bar{N}\right)\left(I - \Theta_{I_\Theta}\right) B^T H_{k+1} B (I - \Theta_{I_\Theta})\left(N_I - \bar{N}\right) +$$
$$+ 2u_k^T \overline{N}\left(I - \Theta_{I_\Theta}\right) B^T S_{k+1} B \Theta_{I_\Theta} N_I u +$$
$$\left.\left. + \overline{N}\left(I - \Theta_{I_\Theta}\right) B^T S_{k+1}\left(I - \Theta_{I_\Theta}\right)\overline{N} B\right)\right]$$

and $\Theta_I = N_I$. Matrices T_k,S_k,D_k remain the same defined in (34),(35),(36) and $H_k = T_k - S_k$.

□

V. Example

This section is devoted to show how the probability of receiving an acknowledgement from the actuators affects the stability regions of the LQG controller. In order to exploit necessary and sufficient conditions arising from equation (40), we consider a very simple system with an invertible and square B:

$$\begin{aligned} x(t+1) &= 3x(t) + u(t) + w(t) \\ y(t) &= x(t) \end{aligned} \tag{53}$$

with $Q = 1$. Figure 3 shows the different stability regions with respect to $\bar{\nu}$ and $\bar{\gamma}$, parameterized by the acknowledgement probability $\bar{\theta}$. In particular it is possible to show that, as $\bar{\theta} \to 1$ the stability region converges to the one computed for the TCP-like protocol.

VI. Conclusions

In this paper we analyzed a generalized version of the LQG control problem for the case where both observation and control packets may be lost during transmission over a communication channel and in which a stochastic input acknowledgment mechanism is provided. We have shown that the partial lack of acknowledgement of control packets results in the failure of the separation principle and that estimation and control are now intimately coupled. We have shown that the LQG optimal control is linear only in the particular case where we have access to full state information. In such a case, the partial presence of acknowledgements increases the stability range of the overall system, converging to the TCP-like with deterministic acknowledgements as the arrival rate for the acknowledgement packets tends to one.

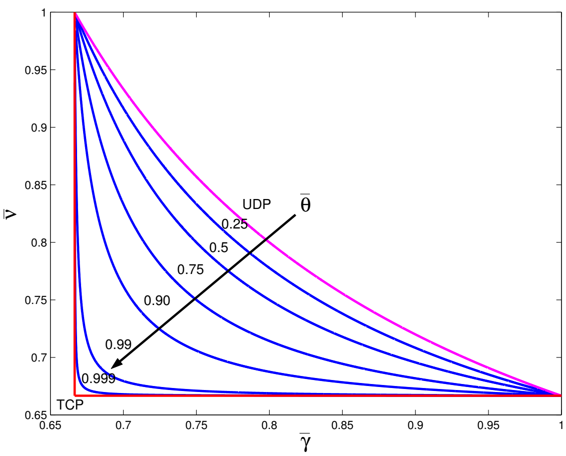

Fig. 3. Region of stability relative to measurement packet arrival probability $\bar{\gamma}$, and the control packet arrival probability $\bar{\nu}$, parameterized into the acknowledgment packet arrival probability $\bar{\theta}$

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