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LQG Control For Distributed Systems Over TCP-like Erasure Channels

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Abstract— **This paper is concerned with control applications over lossy data network. Sensor data is transmitted to an estimation-control unit over a network and control commands are issued to subsystems over the same network. Sensor and control packets may be randomly lost according to a Bernoulli process. In this context the discrete-time Linear Quadratic Gaussian (LQG) optimal control problem is considered. In [1] a complete analysis was carried out for the case the network is composed of a single sensor and control channel. Here a nontrivial generalization to the case of sensor and actuator networks with p distinct sensor channels and m control channels is presented. It has been proven that the separation principle still holds for all protocols where packets are acknowledged by the receiver (e.g. TCP-like protocols). Moreover it has been pointed out for the first time that the optimal LQG control is a linear function of the state that explicitly depends on the command channels lost probabilities. Such a dependence is not present in pre-existing literature, since the amplitude of each control input has to be weighted by the loss probability associated to its own channel. This is not observed in the single channel case. In the infinite horizon case stability conditions on the arrival are derived. Their computation requires the use of Linear Matrix Inequalities (LMIs).**

I. INTRODUCTION

Today, an increasing number of applications demands remote control of plants over unreliable networks. The recent development of sensor web technology [1] enables the development of wireless sensor networks that can be immediately used for estimation and control. In these systems issues of communication delay, data loss and time-synchronization play critical roles. Communication and control become tightly coupled and these two issues cannot be addressed independently. The goal of this paper is to provide some partial answers to the question of how control loop performance is affected by communication constraints and what are the basic system-theoretic implications of using unreliable networks for control. This requires a generalization of classical control techniques that explicitly take into account the stochastic nature of the communication channel.

We consider a generalized formulation of the Linear Quadratic Gaussian (LQG) optimal control problem by modeling the arrival of both observations and control packets as random processes whose parameters are related to the characteristics of the communication channel. We envision a distributed system, where components, such as sensors, actuators and the controller have to communicate wirelessly. Therefore the system will have multiple sensing and control channels. Accordingly, in full generality, multiple independent Bernoulli processes are considered, with parameters $\overline{\gamma}_i$ and $\overline{\nu}_j$, that govern packet losses between the sensors and the estimation-control unit, and between the latter and the actuation points (see Figure 1). In our analysis, we distinguish between two classes of protocols. The distinction resides simply in the availability of packet acknowledgements. Adopting the framework proposed by Imer *et al.* [2], we will refer therefore to TCP-like protocols if packet acknowledgements are available and to UDP-like protocols otherwise.

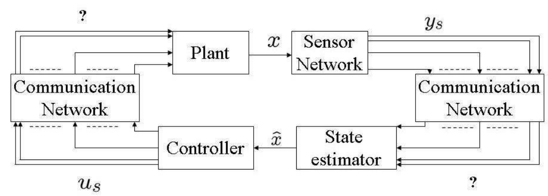

Fig. 1. **Overview of the system.** We study the statistical convergence properties of the expected state covariance of the discrete time LQG control system, when both the observation and the control signal, transmitted over unreliable communication channels, can be lost at each time step in channels i and j with probability $1 - \bar{\gamma}_i$ and $1 - \bar{\nu}_j$ respectively.

Previous results on this topic [3], [4], [5], [6] are summarized in Figure 2. They refer to the case where only a single channel exists between both sensors and controller and between the latter and the actuators. They have shown the existence of a critical domain of values for the parameters of the Bernoulli arrival processes, $\overline{\nu}$ and $\overline{\gamma}$, outside which a transition to instability occurs and the optimal controller fails to stabilize the system. In particular, under TCP-like protocols, the critical arrival probabilities for the control and observation channel are independent of each other. This is another consequence of the fact that the separation principle holds for these protocols. A more involved situation regards UDP-like protocols. In this case the critical arrival probabilities for the control and observation channels are coupled. The stability domain and the performance of the optimal controller degrade considerably as compared with TCP-like protocols as shown in Figure 2.

It was also shown that for the TCP-like case the classic separation principle holds, and consequently the controller and estimator can be designed independently. Moreover, the optimal controller is a linear function of the state. In sharp contrast, for the UDP-like case, the optimal controller is in general non-linear. In this case, a natural sub-optimal solution is to use the optimal static linear gain, explored

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in [6]. This is particularly attractive for sensor networks, where simplicity of implementation and complexity issues are a primary concern.

In this paper we focus on the general case where multiple communication channels are interposed between the sensors, the controller and the actuators. Therefore, contrarily to the previous work, the system can experience partial observation and control loss. In [7] this case was considered for the estimation problem exclusively and with two observation channels. Even in the general case of multiple channels we can prove that the separation principle holds. The optimal estimator is linear as well as the optimal controller. We solve the LQG problem for both the finite and infinite horizon case. For the latter we provide implicit stability conditions.

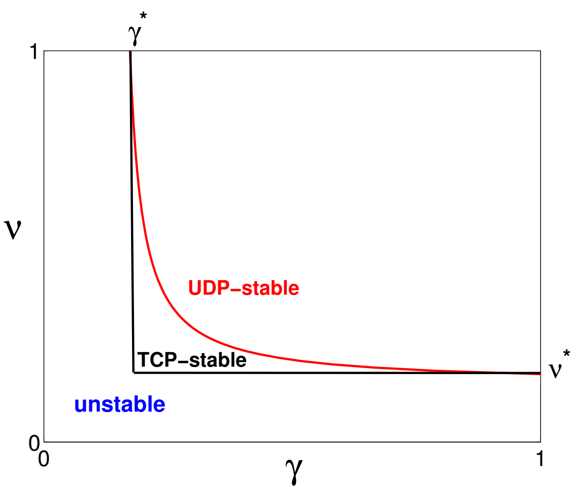

Fig. 2. Region of stability for UDP-like and TCP-like optimal control relative to measurement packet arrival probability γ, and the control packet arrival probability ν.

We now wish to mention some closely related research. Study of stability of dynamical systems where components are connected asynchronously via communication channels has received considerable attention in the past few years and our contribution can be put in the context of the previous literature. In [8] and [9], the authors proposed to place an estimator, i.e. a Kalman filter, at the sensor side of the link without assuming any statistical model for the data loss process. In [10], Smith *et al.* considered a suboptimal but computationally efficient estimator that can be applied when the arrival process is modeled as a Markov chain, which is more general than a Bernoulli process. Drew et al [11] analyze the problem of designing a controller over a wireless LAN. Control design has been investigated in the context of Cross Layer Design by Liu et al [12]. Finally, Elia [13][14] proposed to model the plant and the controller as deterministic time invariant discrete-time systems connected to zero-mean stochastic structured uncertainty. The variance of the stochastic perturbation is a function of the Bernoulli parameters, and the controller design is posed an an optimization problem to maximize mean-square stability of the closed loop system. This approach allows analysis of Multiple Input Multiple Output (MIMO) systems with many different controller and receiver compensation schemes [13], however, it does not include process and observation noise and the controller is restricted to be time-invariant, hence sub-optimal. The remainder of this paper is organized as follows. Section 2 provides the problem formulation. In Section 3 we solve the estimation problem. In Section 4 we consider the control synthesis problem for both finite and infinite horizon. Finally, Section 5 draws conclusions and outlines the agenda for future work.

II. Problem Formulation

Consider the following linear stochastic system with intermittent observation and control packets:

$$
\begin{aligned}
&x_{k+1} = Ax_k + Bu_k^a + \omega_k, \\
&u_k^a = \mathrm{N}_k u_k + \left[I_{m\times m} - \mathrm{N}_k\right] u_k^l, \\
&y\left(k\right) = \Gamma_k C x_k + v_k, \\
&N_k = \begin{bmatrix} \nu_{1,k} & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & \nu_{m,k} \end{bmatrix}, \\
&\Gamma_k = \begin{bmatrix} \gamma_{1,k} & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & \gamma_{p,k} \end{bmatrix} = \begin{bmatrix} g_1^T \\ \dots \\ g_p^T \end{bmatrix},
\end{aligned}
\tag{1}
$$

where $x_k \in \mathcal{R}^n$ is the state vector, $y_k \in \mathcal{R}^p$ is the output vector,$(x_0 \in \mathcal{R}^n, w_k \in \mathcal{R}^n, v_k \in \mathcal{R}^p)$ are Gaussian, uncorrelated, white, with mean $(\overline{x_0}, 0, 0)$ and covariance (P_0, Q, R) respectively, and $(\gamma_{i,k})$, $i = 1, \dots, p$ and $(\nu_{j,k})$ $j = 1, \dots, m, \forall k \in \mathbb{Z}$, are i.i.d. Bernoulli random binary variable modeling the successful transmission of the information on a channel i-th sensor and actuator respectively. The probability of successful transmission is $\overline{\gamma_i} = P\left(\gamma_{i,k} = 1\right), i = 1, \dots, p$ and $\overline{\nu_j} = P\left(\nu_{j,k} = 1\right), i = 1, \dots, m$. $u_k^a \in \mathcal{R}^m$ is the effective control input applied to the actuators while $u_k \in \mathcal{R}^m$ denotes the desired control input computed by the controller. Finally $u_k^l \in \mathcal{R}^m$ is the signal locally provided to the actuators in the case $N_k = 0_{m\times m}$ (all packets to the actuators are lost). While it is possible to choose $u^l\left(k\right)$ in several ways, the most common strategies are the following:

1) zero-input scheme: $u_k^l = 0$
2) hold-input scheme: $u_k^l = u_{k-1}^a$

It is important to define the Information Set the controller is equipped with:

$$
I_k = \begin{cases} F_k = \{y_i, \Gamma_i, \mathrm{N}_{i-1} | i = 0, .., k\} & TCP-like \\ G_k = \{y_i, \Gamma_i | i = 0, .., k\} & UDP-like \end{cases}
$$

Let us define the following cost function:

$$
\begin{aligned}
&J_N\left(u^{N-1}, \overline{x_0}, P_0\right) = \\
&E\left[x_N^T W_N x_N + \sum_{k=0}^{N-1} x_k^T W_k x_k + u_k^{aT} U_k u_k^a | u^{N-1}, \overline{x_0}, P_0\right],
\end{aligned}
\tag{2}
$$

the goal is to compute an optimal control input sequence $u^*\left(\cdot\right) = f\left(k, I\left(k\right)\right)$ such that it minimizes the above functional, i.e.:

$$\min_{u_k = f_k(I_k)} J_N\left(u^{N-1}, \overline{x_0}, P_0\right)$$

III. Optimal Estimation

This section is devoted to the computation of the optimal state estimator for a multichannel system under TCP-like protocols. In the first subsection some definition and some previously published results are reported, then in the second one the Optimal Estimator under TCP-like protocols is derived as a convenient generalization of the results in [15] and [7] to the multichannel case.

A. Mathematical Preliminaries

Let us define the following variables:

$$\begin{aligned} \hat{x}_{k|k} &\triangleq E[x_k|I_k] \\ e_{k|k} &\triangleq x_k - \hat{x}_{k|k} \\ P_{k|k} &\triangleq E[e_k e_k^T|I_k]. \end{aligned} \quad (3)$$

The derivations below will make use of the following lemma:
Lemma 1 - The following equalities hold true:

$$E\left[e_{k|k}\hat{x}_{k|k}^T|I_k\right] = 0 \quad (4)$$

$$E\left[x_k^T S x_k|I_k\right] = \hat{x}_k^T S\hat{x}_k + trace\left(SP_{k|k}\right), \forall S \geq 0 \quad (5)$$

$$E\left[E\left[f\left(x_{k+1}\right)|I_{k+1}\right]|I_k\right] = E\left[f\left(x_{k+1}\right)|I_k\right], \forall f. \quad (6)$$

Proof - See [3]. □

In order to derive optimal LQG controller it is important to know the expected value of $x_k^T\, S\, x_k$. By exploiting the independence of N_k, w_k, x_k and the zero-mean property of w_k it is possible to prove that:

$$\begin{aligned} &E\left[x_{k+1}Sx_{k+1}|I_k\right] = E\left[\left(Ax_k\right)^T S\left(Ax_k\right)|I_k\right] + \\ &+2\left(B\overline{N}u_k + B\left(I-\overline{N}\right)u_k^l\right)^T SA\hat{x}_{k|k} + trace\left(SQ\right) \\ &+ \sum_{I\in 2^{\Im}}\Bigg[\left(\prod_{i\in I}\overline{\nu_i}\right)\left(\prod_{i\notin I}\left(1-\overline{\nu_i}\right)\right)\left(BN_I u_k + B\left(I-N_I\right)u_k^l\right)^T \\ &\qquad S\left(BN_I u_k + B\left(I-N_I\right)u_k^l\right)\Bigg]. \end{aligned} \quad (7)$$

In the *zero-input scheme* this becomes:

$$\begin{aligned} &E\left[x_{k+1}^T Sx_{k+1}|I_k\right] = E\left[\left(x_k^T A^T TSAx_k\right)|I_k\right] + \\ &+ \sum_{I\in 2^{\Im}}\left[\left(\prod_{i\in I}\overline{\nu_i}\right)\left(\prod_{i\notin I}\left(1-\overline{\nu_i}\right)\right)u_k^T N_I B^T SBN_I u_k\right] \\ &+ 2u_k^T\overline{N}B^T SA\hat{x}_{k|k} + trace\left(SQ\right), \end{aligned} \quad (8)$$

where $\overline{N} = diag\{\overline{\nu}_1, ..., \overline{\nu}_m\}$. N_I is a diagonal matrix defined on the index set $I \subseteq \Im = \{1, ..., m\}$ such that

$$(N_I)_{ii} = \begin{cases} 1, if \;\; i \in I\} \\ 0, if \;\; i \notin I\} \end{cases}$$

and $2^{\Im}$ is the set of all the possible subsets of $\Im$.
At last it is important to notice that

$$\begin{aligned} E\left[e_{k|k}^T Te_{k|k}|I_k\right] &= trace\left(TE\left[e_{k|k}^T e_{k|k}|I_k\right]\right) \\ &= trace\left(TP_{k|k}\right). \end{aligned} \quad (9)$$

B. Optimal Estimator under TCP-like protocols

The equation for the optimal estimator can be derived in the case of partial observation losses [7] using similar arguments to the ones used in the standard Kalman filter. The innovation step is the following:

$$\hat{x}_{k+1|k} = AE\left[x_k|F_k\right] + BN_k u_k = A\hat{x}_{k|k} + BN_k u_k, \quad (10)$$

$$e_{k+1|k} = Ae_{k|k} + w_k, \quad (11)$$

$$P_{k+1|k} = AP_{k|k}A^T + Q, \quad (12)$$

where the independence of w_k and F_k and the fact that $N_k u_k$ is a deterministic function of F_k are exploited. In order to obtain a more compact correction step w.r.t. the combinatorial one proposed in [7], let us introduce the following rows selection matrix

$$\Gamma_k^m = \left[g_i^T\right]_{\gamma_{i,k}=1},$$

i.e. the matrix of the nonzero rows of Γ_k. Using this representation it is finally possible to show that the correction step is:

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1}\Gamma_{k+1}^m\left(y_{k+1} - C\hat{x}_{k+1|k}\right), \quad (13)$$

$$K_{k+1} = P_{k+1|k}C^T\Gamma_{k+1}^{mT}\left(\Gamma_{k+1}^m\left(CP_{k+1|k}C^T + R\right)\Gamma_{k+1}^{mT}\right)^{-1}, \quad (14)$$

$$P_{k+1|k+1} = P_{k+1|k} - K_{k+1}\Gamma_{k+1}^m CP_{k+1|k}. \quad (15)$$

Remark 1 - Note that such a formulation of the Kalman filter for sensor networks can be seen as a time varying Kalman filter using the following time varying system matrices: $A_k = A, C_k = \Gamma_k^m C$ where $C_k \in \mathcal{R}^{p_k \times n}$ and p_k is the number of nonzero $\gamma_{i,k}$ at time k, i.e. the number of received packets at time k. The converges properties of the filter are studied in Section IV, where proofs are omitted due to space constraint. For a complete treatment please refer to [16]. □

IV. Optimal controller

In this section we analyze the LQG control design problem for the system(1). Both the finite and infinite horizon cases are considered.

A. Finite horizon LQG Control

In order to derive the optimal control law and the corresponding value for the objective function we follow the dynamic programming approach based on the cost-to-go iterative procedure. Let us define the optimal value function $V_k(x_k)$ as follows:

$$V_N\left(x_N\right) = E\left[x_N^T W_N x_N|F_N\right], \quad (16)$$

$$\begin{aligned} &V_k\left(x_k\right) = \\ &\min_{u_k} E\left[x_k^T W_k x_k + u_k N_k^T U_k N_k u_k + V_{k+1}\left(x_{k+1}\right)|F_k\right], \end{aligned} \quad (17)$$

where $k = N-1, \ldots, 1$. Using dynamic programming theory [17], one can show that $J_N = V_0(x_0)$. Under TCP-like protocols it is possible to prove the next lemma.

Lemma 1 - The value function $V_k(x_k)$ defined in (16)-(17) for the system (1) under TCP-like protocols can be written as

$$V_k(x_k) = E\left[x_k^T S_k x_k | F_k\right] + c_k \quad k = N, ..., 0. \tag{18}$$

The matrix S_k and the scalar c_k can be computed in the following way:

$$S_k = W_k + A^T S_{k+1} A - A^T S_{k+1} B \overline{N} \left[\sum_{I\in 2^{\Im}} \left(\prod_{i\in I} \overline{\nu_i}\right)\left(\prod_{i\notin I} 1 - \overline{\nu_i}\right)\left(\mathrm{N}_I \left(U_k + B^T S_{k+1} B\right) \mathrm{N}_I\right)\right]^{-1} \overline{N} B^T S_{k+1} A, \tag{19}$$

$$c_k = E\left[c_{k+1}|F_k\right] + trace\left(S_{k+1}Q\right) + +trace\left(\left(A^T S_{k+1} A + W_k - S_k\right) P_{k|k}\right), \tag{20}$$

with initial values S_N=W_N and c_N=0. Moreover the optimal control input is given by:

$$u_k = -\left[\sum_{I\in 2^{\Im}} \left(\prod_{i\in I} \overline{\nu_i}\right)\left(\prod_{i\notin I} (1 - \overline{\nu_i})\right)\left(\mathrm{N}_I \left(U_k + B^T S_{k+1} B\right) \mathrm{N}_I\right)\right]^{-1} \left[\overline{N} B^T S_{k+1} A\right] x_{k|k} = L_k x_{k|k}. \tag{21}$$

Proof - The proof employs an induction argument. The claim is clearly true for $k = N$ with parameter $S_N = W_N$ and $c_N = 0$. Suppose now that the claim is true for $k+1$ and so $V_{k+1}(x_k) = E[x_{k+1}^T S_{k+1} x_{x+1} | F_{k+1}] + c_{k+1}$. The cost at time k is:

$$\begin{aligned}
&V_k(x_k) = \\
&\min_{u_k} E\left[x_k^T W_k x_k + u_k^T \mathrm{N}_k U_k \mathrm{N}_k u_k + V_{k+1}(x_{k+1}) | F_k\right] = \\
&= \min_{u_k}(E\left[x_k^T W_k x_k + u_k^T \mathrm{N}_k U_k \mathrm{N}_k u_k | F_k\right] + \\
&E\left[E\left[x_{k+1}^T S_{k+1} x_{k+1} + c_{k+1} | F_{k+1}\right] | F_k\right]) \\
&= \min_{u_k} E\left[x_k^T W_k x_k + u_k^T \mathrm{N}_k U_k \mathrm{N}_k u_k + x_{k+1}^T S_{k+1} x_{k+1} + c_{k+1} | F_k\right] \\
&= \min_{u_k}(E[x_k^T W_k x_k + u_k^T \mathrm{N}_k U_k \mathrm{N}_k u_k + (Ax_k + B\mathrm{N}_k u_k + \omega_k)^T \\
&S_{k+1}(Ax_k + B\mathrm{N}_k u_k + \omega_k) + c_{k+1} | F_k])
\end{aligned}$$

Finally we obtain:

$$\begin{aligned}
&V_k(x_k) = \min_{u_k}(E\left[x_k^T W_k x_k + x_k^T A^T S_{k+1} A x_k + c_{k+1} | F_k\right] + \\
&+E[u_k^T \mathrm{N}_k U_k \mathrm{N}_k u_k + u_k^T \mathrm{N}_k B^T S_{k+1} B \mathrm{N}_k u_k + \omega_k^T S_{k+1} \omega_k \\
&+2u_k^T N_k B^T S_{k+1} A x_k | F_k]) = \\
&= \min_{u_k}(E\left[x_k^T W_k x_k + x_k^T A^T S_{k+1} A x_k | F_k\right] + E\left[c_{k+1}|F_k\right] \\
&+trace\left(S_{k+1}Q\right) + E[u_k^T \mathrm{N}_k \left(U_k + B^T S_{k+1} B\right) \mathrm{N}_k u_k \\
&+2u_k^T N_k B^T S_{k+1} A x_k | F_k]) = \\
&= \min_{u_k}\Bigg(E\left[x_k^T W_k x_k + x_k^T A^T S_{k+1} A x_k | F_k\right] + \left[c_{k+1}|F_k\right] \\
&+trace\left(S_{k+1}Q\right) + 2u_k^T \overline{N} B^T S_{k+1} A x_{k|k} \\
&+\sum_{I\in 2^{\Im}}\left[\left(\prod_{i\in I} \overline{\nu_i}\right)\left(\prod_{i\notin I}(1 - \overline{\nu_i})\right)\left(u_k^T \mathrm{N}_I \left(U_k + B^T S_{k+1} B\right) \mathrm{N}_I u_k\right)\right]\Bigg)
\end{aligned}$$

Since the value function is a quadratic function of the input, then the minimizer can be obtained by solving $\partial V_k / \partial u_k = 0$:

$$\frac{\partial V_k(x_k)}{\partial u_k} = 2\overline{N} B^T S_{k+1} A x_{k|k} + \sum_{I\in 2^{\Im}}\left[\left(\prod_{i\in I} \overline{\nu_i}\right)\left(\prod_{i\notin I}(1 - \overline{\nu_i})\right)\left(2\mathrm{N}_I \left(U_k + B^T S_{k+1} B\right) \mathrm{N}_I u_k\right)\right] = 0,$$

which yields equation (21).The optimal law is thus a linear function of the state estimate. If we substitute now the minimizer back into the cost $V_k(x_k)$ we obtain the following expression:

$$\begin{aligned}
&V_k(x_k) = E\left[x_k^T W_k x_k + x_k^t A^T S_{k+1} A x_k | F_k\right] + \\
&E\left[c_{k+1}|F_k\right] + trace\left(S_{k+1}Q\right) - x_{k|k}^T A^T S_{k+1} B T \overline{N} \\
&\left[\sum_{I\in 2^{\Im}} \left(\prod_{i\in I} \overline{\nu_i}\right)\left(\prod_{i\notin I}(1 - \overline{\nu_i})\right)\left(\mathrm{N}_I \left(U_k + B^T S_{k+1} B\right) \mathrm{N}_I\right)\right]^{-1} \\
&\overline{N} B^T S_{k+1} A x_{k|k}
\end{aligned}$$

Finally, using (5) of Lemma 1, we can rewrite $V_k(x_k)$ as

$$\begin{aligned}
&V_k(x_k) = E\Bigg[x_k^T W_k x_k + x_k^T A^T S_{k+1} A x_k - x_k^T A^T S_{k+1} B T \overline{N} \\
&\left[\sum_{I\in 2^{\Im}} \left(\prod_{i\in I} \overline{\nu_i}\right)\left(\prod_{i\notin I}(1 - \overline{\nu_i})\right)\left(\mathrm{N}_I \left(U_k + B^T S_{k+1} B\right) \mathrm{N}_I\right)\right]^{-1} \\
&\overline{N} B^T S_{k+1} A x_k | F_k\Bigg] + E\left[c_{k+1}|F_k\right] + trace\left(S_{k+1}Q\right) \\
&+trace\Bigg(A^T S_{k+1} B \overline{N}\left[\sum_{I\in 2^{\Im}} \left(\prod_{i\in I} \overline{\nu_i}\right)\left(\prod_{i\notin I}(1 - \overline{\nu_i})\right)\right. \\
&\left.\left(\mathrm{N}_I \left(U_k + B^T S_{k+1} B\right) \mathrm{N}_I\right)\right]^{-1} \overline{N} B^T S_{k+1} A P_{k|k}\Bigg).
\end{aligned}$$

Using this last equation we can easily see how equation (18) is satisfied also for the step k for all x_k if and only if the matrices S_k and the scalars c_k satisfy equations (19) and (20) respectively. □

As a consequence of the above result it follows that, since $J_N^*(x_0, P_0) = V_0(x_0)$, the cost function for the LQG control problem under TCP-like protocols is given by:

$$J_N^* = \overline{x}_0^T S_0 \overline{x}_0 + trace\left(S_0 P_0\right) + \sum_{k=0}^{N-1} trace\left(S_{k+1}Q\right) + \sum_{k=0}^{N-1} trace\left(\left(A^T S_{k+1} A + W_k - S_k\right) E_\Gamma\left[P_{k|k}\right]\right). \tag{22}$$

The following theorem summarizes the results for the finite horizon LQG control under TCP-like communication protocols for distributed networked systems:

Theorem (Finite Horizon LQG under TCP) - Consider the system (1) and consider the problem of minimizing the cost function (18) with policy $u_k = f(I_k)$ where I_k is the information available under TCP like communication protocol. Then, the optimal cost is a **linear** function of the estimated system state (21), where the matrix S_k can be computed iteratively using (19). The **separation principle** still holds under TCP-like communication, since the optimal estimator is independent of the control input u_k. The optimal state estimator is given by (10)-(15) and the minimal achievable cost is given by (22). □

Remark 2 - It is important to notice how, in the multichannel case, the optimal control depends directly from the arrival rates of every single control channel. This is a consequence of the fact that the control inputs are weighed by the arrival rate relative to their own channel. □

Remark 3 - It is worth remarking that the error covariance matrices $\{P_{k|k}\}_{k=0}^{N}$ are stochastic since they depend on the sequence $\{\Gamma_k\}$. Since $P_{k+1|k+1}$ is a nonlinear difference equation the exact expected value of matrices $E_\Gamma[P_{k|k}]$ cannot be computed analytically even in the single-channel case. However, they can be bounded by deterministic quantities. For more details please refer to [15] and [7]. □

B. Infinite Horizon LQG Control

The infinite horizon LQG can be obtained by taking the limit for $N \to \infty$ of the previous equations. However, as explained in Remark 3, the matrices $\{P_{k|k}\}$ depend nonlinearly on the specific realization of the matrix observation sequence $\{\Gamma_k\}$, therefore the expected covariance matrices $E_\Gamma[P_{k|k}]$ and the minimal cost J_N^* cannot be computed analytically and do not seem to have limit. Moreover it is important to understand that, differently from the standard LQG regulator, in the case of observation and control packet losses, the stability can be lost if the arrival probabilities are below a certain threshold. In order to analyze this behavior we need to study the following Modified Algebraic Riccati Equations (MAREs) for both the controller and the estimator respectively:

$$S = \Pi_c\left(S, A, B, Q, R, \overline{N}\right) \tag{23}$$

$$P = \Pi_o\left(P, A, C, Q, R, \overline{\Gamma}\right), \tag{24}$$

where the nonlinear operators Π_c,Π_o are defined as follows:

$$\Pi_c\left(S, A, B, Q, R, \overline{N}\right) = W + A^TSA - A^TSB\overline{N} \left[\sum_{I\in 2^{\Im}}\Big(\prod_{i\in I}\overline{\nu_i}\Big)\Big(\prod_{i\notin I}1-\overline{\nu_i}\Big)\left(\mathrm{N}_I\left(U + B^TSB\right)\mathrm{N}_I\right)\right]^{-1} \overline{N}B^TSA, \tag{25}$$

$$\Pi_o\left(P, A, C, Q, R, \overline{\Gamma}\right) = A^TPA + Q - A^TPC^T \sum_{I\in 2^{\Im}}\left[\Big(\prod_{i\in I}\overline{\gamma_i}\Big)\Big(\prod_{i\notin I}(1-\overline{\gamma_i})\Big)\Gamma_I^{mT}\left(\Gamma_I^m\left(CPC^T + R\right)\Gamma_I^{mT}\right)^{-1}\Gamma_I^m\right]CPA. \tag{26}$$

Moreover let us define the following operators:

$$\phi_c(K, X) = \left[\sum_{I\in 2^{\Im}}\Big(\prod_{i\in I}\overline{\nu_i}\Big)\Big(\prod_{i\notin I}1-\overline{\nu_i}\Big)\left(F_I^cXF_I^c + V_I^c\right)\right], \tag{27}$$

$$\phi_o(K_\emptyset, \dots, K_\Im, X) = \left[\sum_{I\in 2^{\Im}}\Big(\prod_{i\in I}\overline{\gamma_i}\Big)\Big(\prod_{i\notin I}1-\overline{\gamma_i}\Big)\left(F_I^oXF_I^o + V_I^o\right)\right], \tag{28}$$

with:

$$F_I^c \triangleq A^T + K(N_IB^T) \tag{29}$$

$$V_I^c \triangleq W + KN_IUN_I^TK^T \tag{30}$$

$$F_I^o \triangleq A + K_I(\Gamma_I^mC) \tag{31}$$

$$V_I^o \triangleq Q + K_I\Gamma_I^mR\Gamma_I^{mT}K_I^T \tag{32}$$

and where $\emptyset, \dots, \Im$, denotes the enumeration of any $I \in 2^{\Im}$. It is now possible to state the following results.

Theorem (Convergence of MAREs) - If there exists a pair $(\tilde{K}_c, \tilde{S})$ such that

$$\tilde{S} > \phi_c(\tilde{K}_c, \tilde{S}), \;\; \tilde{S} > 0 \tag{33}$$

then, for any initial condition $S_0 > 0$, the MARE (23) converges and the limit is independent of the initial condition i.e. $lim_{t\to\infty}S_t = \overline{S}$ where $\overline{S}$ is the unique positive-semidefinite fixed point of the MARE. Similarly, if there exists a (2^p+1)-tuple such that

$$\tilde{P} > \phi_o(\tilde{K}_\emptyset, \dots, \tilde{K}_\Im, \tilde{P}), \;\; \tilde{P} > 0 \tag{34}$$

then, for any initial condition $P_0 > 0$, the MARE (24) converges to $\overline{P}$, the unique positive-semidefinite fixed point of the MARE. □

Lemma 2 - Conditions (33) and (34) are equivalent to the solution of the two following LMIs feasibility problems

$$\Psi_c(Y, Z) = \begin{pmatrix} Y & Y & \eta_\emptyset\left(YA^T + ZN_\emptyset B\right) & \eta_\emptyset KN_\emptyset U^{1/2} & \dots & \eta_\Im\left(YA^T + ZN_\Im B\right) & \eta_\Im KN_\Im U^{1/2} \\ \dots & W^{-1} & 0 & 0 & \dots & 0 & 0 \\ * & \dots & Y & 0 & \dots & 0 & 0 \\ * & * & \dots & I & \dots & 0 & 0 \\ * & * & * & \dots & \dots & \dots & \dots \\ * & * & * & * & \dots & Y & 0 \\ * & * & * & * & \dots & * & I \end{pmatrix} > 0, \quad 0 < Y \le I, \tag{35}$$

and

$$\Psi_o(Y, Z_\emptyset, \dots, Z_\Im) = \begin{pmatrix} Y & Y & \lambda_\emptyset\left(YA + Z_\emptyset\Gamma_\emptyset^mC\right) & \lambda_\emptyset Z_\emptyset\Gamma_\emptyset^m R^{1/2} & \dots & \lambda_\Im\left(YA + Z_\Im\Gamma_\Im^mC\right) & \lambda_\Im Z_\Im\Gamma_\Im^m R^{1/2} \\ \dots & W^{-1} & 0 & 0 & \dots & 0 & 0 \\ * & \dots & Y & 0 & \dots & 0 & 0 \\ * & * & \dots & I & \dots & 0 & 0 \\ * & * & * & \dots & \dots & \dots & \dots \\ * & * & * & * & \dots & Y & 0 \\ * & * & * & * & \dots & * & I \end{pmatrix} > 0, \quad 0 < Y \le I, \tag{36}$$

where

$$\eta_I = \sqrt{\Big(\prod_{i\in I}\overline{\nu_i}\Big)\Big(\prod_{i\notin I}1-\overline{\nu}_i\Big)}, \;\; \lambda_I = \sqrt{\Big(\prod_{i\in I}\overline{\gamma_i}\Big)\Big(\prod_{i\notin I}1-\overline{\gamma_i}\Big)}.$$

Proof - See [16] for a complete proof. □

Theorem (Infinite Horizon LQG under TCP) - Consider the system (1) under the following additional hypothesis: $W_N = W_k = W$ and $U_k = U$. Moreover, let (A, B) and $(A, Q^{1/2})$ be controllable and (A, C) and $(A, W^{1/2})$ be observable, then, if the arrival probabilities $\overline{N}$,$\overline{\Gamma}$ are such that MAREs (23)-(24) converges, then the infinite horizon optimal controller gain is constant:

$$L_\infty = -\left[\sum_{I\in 2^{\Im}} \left(\prod_{i\in I} \overline{\nu_i}\right)\left(\prod_{i\notin I}(1-\overline{\nu_i})\right)\left(\mathrm{N}_I\left(U_k + B^T S_\infty B\right)\mathrm{N}_I\right)\right]^{-1} \left[\overline{N} B^T S_\infty A\right]. \quad (37)$$

The infinite horizon optimal estimator gain L_k, given by equation (14), is time-varying since it depends on the realization of the observation arrival process $\{\Gamma_j\}_{j=1}^k$. □

Remark 4 - In the infinite horizon case of the multichannel model, transition from instability to boundedness of the state depends on the arrival rates of each channel. While the LMI grows exponentially with the number of channel, this is an operation which needs to be carried out only once at design time. All the proofs were omitted for lack of space. For a complete characterization refer to [16]. □

V. Conclusion and Future Work

Motivated by applications where control is performed over a communication network, this paper extends previous results on optimal control over lossy networks to the case where both observation and control packets travel across a multichannel network. As a consequence partial observation and control input losses may occur. We assume that an acknowledgement of the arrival of the control packet is always available to the controller (TCP). First, we computed the optimal estimator for this case. Then we proved that the optimal LQG control is a linear function of the state, showing that the separation principle also holds. We computed the optimal controller for both finite and infinite horizon, providing stability conditions for the infinite horizon case. Future work will involve the analysis for the case when the controller does not receive any acknowledgement to whether its packet has been received by the actuator or not.

VI. References

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