Back

An off-line MPC strategy for nonlinear systems based on SOS programming

Giuseppe Franzè[†], Alessandro Casavola[†], Domenico Famularo[‡] and Emanuele Garone[†]

[†] *DEIS - Università degli studi della Calabria, Rende (CS), 87036, ITALY,* {casavola,franze,egarone}@deis.unical.it
[‡] *DIMET - Università degli studi di Reggio Calabria, Via Graziella, Loc. Feo di Vito, Reggio Calabria (RC), 89100, ITALY,* domenico.famularo@unirc.it

Keywords : Sum of Squares, Nonlinear Systems, Predictive Control, Convex Relaxations, Constrained Systems

Abstract : A novel moving horizon control strategy for input-saturated nonlinear polynomial systems is proposed. The control strategy makes use of the so called sum-of-squares (SOS) decomposition, i.e. a convexification procedure able to give rise sufficient conditions on the positiveness of polynomials. The complexity of SOS-based numerical methods is polynomial in the problem size and, as a consequence, computationally attractive. SOS programming is used here to derive an "off-line" model predictive control (MPC) scheme and analyze in depth its relevant properties. The main contribution here is to show that such an approach may lead to less conservative MPC strategies than most existing methods based on global linearization approaches. An illustrative example is provided to show the effectiveness of the proposed SOS-based algorithm.

1 Introduction

Model Predictive Control (MPC) is an optimization based control strategy able to efficiently deal with plant constraints. At each time interval, the MPC algorithm computes an open-loop sequence of inputs by minimizing, compatibly with prescribed constraints, a cost index based on future plant predictions. The first input of the optimal sequence is applied to the plant and the entire optimization procedure is repeated at future time instants.

Though almost all processes are inherently nonlinear, the vast majority of MPC applications and results are based on linear or uncertain linear dynamic models (see [1, 2] and references therein). One of the main reasons for this choice is probably related to the huge on-line computational burdens typically resulting from direct nonlinear programming techniques which are, in some cases, non-convex programming algorithms [3, 4, 5].

Nevertheless, there are cases when nonlinear effects are significant enough to justify the use of direct nonlinear MPC (NMPC) technologies, contrasted to linearized MPC approaches (see [6, 7]). These include at least two broad categories of applications: regulation problems, where the plant is highly nonlinear and subject to large frequent disturbances, and servo problems, where the set point changes frequently and spans a sufficiently wide range of nonlinear process dynamics.

Int. Workshop on Assessment and Future Directions of NMPC
Pavia, Italy, September 5-9, 2008

The purpose of this paper is to consider a particular class of nonlinear plants and constraints described by means of polynomials. The formulation of the MPC problem in such a case gives rise to polynomial optimization problems solvable by using efficient numerical methods exploiting Gröbner bases, cylindrical algebraic decomposition etc., which have been recently proposed in the literature (see [8, 9, 6, 10]).

In particular, SOS decomposition and semidefinite programming [11, 12, 13, 14] techniques will be used here, whose computational complexity is polynomial in the problem size. Strictly speaking, the SOS-based approach is a powerful convexification method which generalizes the well-known S-procedure [9] by searching for polynomial multipliers. As one of its major merits, the SOS-based approach provides less conservative results than most available methods. Preliminary results along this research line have been achieved in [13] where an on-line constrained MPC strategy for open-loop stable polynomial systems has been developed.

Here we propose an off-line formulation of a Receding Horizon Control (RHC) problem for polynomial systems based on the computation of a nested sequence of positive invariant sets (see [15] where a similar algorithm is detailed for uncertain linear plants). With this off-line approach, the SOS computation time is not a limiting factor and increased control performance can be achieved also for fast processes and large scale nonlinear systems.

2 Problem formulation

Consider the following nonlinear system with polynomial vector field

$$x(t+1) \quad = \quad f(x(t)) + g(x(t))\,u(t) \tag{1}$$

where $f \in \mathbb{R}^n[x],\ g \in \mathbb{R}^{n\times m}[x]$, with $x \in \mathbb{R}^n$, denoting the state and $u \in \mathbb{R}^m$ the control input which is subject to the following component-wise saturation constraints

$$u(t) \in \mathcal{U}\,,\ \forall t \geq 0: \quad \mathcal{U} \triangleq \{u \in \mathbb{R}^m \mid |u_i| \leq \bar{u}_i,\ i = 1,\ldots,m,\}\,. \tag{2}$$

here $\mathcal{U}$ is a compact subset of $\mathbb{R}^m$ containing the origin as an interior point. It is assumed in this paper that f and g are continuous and $0_x \in \mathbb{R}^n$ is an equilibrium point for (1) with $u = 0$ [4], i.e. $f(0) = 0$.

The aim is to find, given a certain initial state $x(0)$, a state feedback regulation strategy $u(t) = g(x(t))$ which, under the prescribed constraints (2), asymptotically stabilizes (1) and minimizes the infinite horizon quadratic cost

$$J(u,x(0)) \triangleq \sum_{t=0}^{\infty} \left(\|x(t)\|^2_{\Psi_x} + \|u(t)\|^2_{\Psi_u} \right) \tag{3}$$

where (3) $\Psi_x = \Psi_x^T,\ \ \Psi_u = \Psi_u^T$ are positive definite weighting matrices.

In what follows, a Receding Horizon Control (RHC) scheme for the proposed regulation problem will be introduced and outlined. Therefore, by resorting to well-established ideas (see [1] for a comprehensive and detailed discussion), we look for a guaranteed cost, input constrained state feedback regulation strategy.

We recall that sufficient conditions guaranteeing the feasibility and closed-loop stability of the RHC paradigm for nonlinear discrete-time systems have been presented in [16]. There, under mild assumptions, it has been proved that the RHC optimization problem has a solution if a non-increasing Lypaunov function $V(x(\cdot))$, can be found

(Fundamental Theorem, pag. 293, [16]) and the regional stability of the feedback control system is achieved. Therefore, under a state-dependent control law $u(t) = K(x(t))$, with $x(t)$ the initial state, an upper bound to the quadratic performance index (3) of the form

$$J(K(x(t)),x(t)) \leq V(x(t)) \tag{4}$$

can be found if $V(x)$ is a nonincreasing Lyapunov function. Moreover, $\mathcal{E} := \{x \in \mathbb{R}^n | V(x) \leq \gamma, \ \ \gamma \geq 0\}$, is a positive invariant region for the regulated input constrained system. In the presence of input constraints $u \in \mathcal{U}$, all of the above results continue to be true provided that the pair $(V(x), K(x))$ is chosen so that $\forall x \in \mathcal{E} \Rightarrow K(x) \subset \mathcal{U}$.

3 Main Result

In order to develop a constrained RHC strategy for the nonlinear system (1), the following state-dependent feedback law will be adopted hereafter

$$u(t) = K(x(t)), \ \forall t. \tag{5}$$

where $K(x)$ denotes a multivariate polynomial in the unkwnown x. Moreover it is supposed that the upper-bound $V(x(t))$ to the cost (3), introduced in (4), will be a SOS. Using this assumption and exploiting standard Hamilton-Jacobi-Bellman (HJB) inequality arguments [16], it is possible to derive conditions under which there exists a closed-loop stabilizing control law (5) which achieves a guaranteed cost (4) and is compatible with the input constraints (2):

Proposition 1 *Let $x(t)$ be the current state of the polynomial nonlinear system (1) subject to (2). Then, there exists a state feedback control law of the form (5) ensuring asymptotical stability and constraints fulfilment from t onward if a SOS $V \in \Sigma[x]$, a polynomial $K \in R[x]$ and a scalar $\gamma \geq 0$ are found such that*

- $V(x) > 0 \ \forall x \in \mathcal{R}^n \setminus \{0\}$ *and* $V(0) = 0$;
- *the following Hamilton-Jacobi-Bellman inequality holds true*

$$\{x \in \mathcal{R}^n | V(x) \leq \gamma\} \subseteq \{x \in \mathcal{R}^n | V(x(t+1)) - V(x(t)) + x^T \Psi_x x + K(x)^T \Psi_u K(x) < 0\}$$

- *the saturation constraints are fulfilled*

$$\{x \in \mathcal{R}^n | V(x) \leq \gamma\} \subseteq \{x \in \mathcal{R}^n | K_i(x) \leq \bar{u}_i\}$$

$$\{x \in \mathcal{R}^n | V(x) \leq \gamma\} \subseteq \{x \in \mathcal{R}^n | K_i(x) \geq -\bar{u}_i\}$$

- *the initial state belongs to the positive invariant set $\mathcal{E}$ with margin ε*

$$\{x \in \mathcal{R}^n | (x - x(0))^T (x - x(0)) \leq \varepsilon\} \subseteq \{x \in \mathcal{R}^n | V(x) \leq \gamma\}, \ \varepsilon > 0$$

Proof. See [16]. □

By resorting to a "Positivstellensatz" (P-satz) [17] argument, the conditions stated in Proposition 1 can be recast as the ones of finding a SOS $V \in \Sigma[x]$, a polynomal $K \in R[x]$

and a scalar $\gamma \geq 0$ such that the following set, achieved as the intersection of each single region in the following list, is empty

$$\begin{cases} \{x \in \mathcal{R}^n | V(x) \leq 0, l_1 \neq 0\} \\ \left\{ \ x \in \mathcal{R}^n | V(x) \leq \gamma, l_2 \neq 0, V(x(t+1)) - V(x(t)) + x^T \Psi_x x + K(x)^T \Psi_u K(x) \geq 0 \ \right\} \\ \{x \in \mathcal{R}^n | K_i(x) > \bar{u_i}, V(x) \leq \gamma\} \\ \{x \in \mathcal{R}^n | K_i(x) < -\bar{u_i}, V(x) \leq \gamma\} \\ \{x \in \mathcal{R}^n | (x - x(0))^T (x - x(0)) \geq \varepsilon, V(x) > \gamma\} \end{cases} \tag{6}$$

where $l_1,\ l_2 \in R[x]$ are appropriate positive definite polynomials such that $l_1(0_x) = 0, l_2(0_x) = 0$. Finally, the following result can be stated

Proposition 2 *Let $x(0) \in \mathbb{R}^n$ be given. Then, a pair $(V(x), K(x))$ compatible with the conditions of Proposition 1 and minimizing the upper-bound (4) can be found by solving the following minimization problem:*

$$\min_{V, s_1, s_2, s_{3,i}, s_{4,i}, s_5 \in \Sigma[x],\ K \in \mathcal{R}[x],\ \gamma \geq 0} \gamma$$

subject to

$$V - l_1 \in \Sigma[x] \tag{7}$$

$$-((\gamma - V)s_1 + (V(f(x, K(x))) - V(x) + x^T \Psi_x x + K(x)^T \Psi_u K(x))s_2 + l_2) \in \Sigma[x] \tag{8}$$

$$(\bar{u_i} - K_i) - (\gamma - V)s_{3,i} \in \Sigma[x],\ i = 1, \ldots, m \tag{9}$$

$$(\bar{u_i} + K_i) - (\gamma - V)s_{4,i} \in \Sigma[x],\ i = 1, \ldots, m \tag{10}$$

$$-\varepsilon + (x - x(0))^T (x - x(0))s_5 + (\gamma - V) \in \Sigma[x] \tag{11}$$

under the following conditions on the degrees of the involved polynomials necessary for problem solvability

$$\begin{cases} \max\left(\partial (V s_1), \partial (V s_2)\right) \geq \\ \geq \max\left(\partial (V(f(x), K(x))s_2), \partial \left(x^T \Psi_x x s_2\right), \partial \left(K(x)^T \Psi_u K(x) s_2\right)\right), \\ \partial (V) = \partial (l_1),\ \partial (V s_{3,i}) \geq \partial (K_i),\ i = 1, \ldots, m,\ \ \partial (s_5) + 2 \geq \partial (V) \end{cases} \tag{12}$$

Proof - It can be determined via S-procedure arguments, which are used to remove the inclusions inside (6), and standard SOS considerations, used to remove some polynomial multipliers (see [8], [9], [14]). □

Remark 1 - Note that the decision polynomials $s_1,\ s_2,\ s_{3,i},\ s_{4,i},\ s_5 \in \Sigma[x],\ \ i = 1, \ldots, m$ do not enter linearly in the constraints. Therefore, from a computational point of view, the problem is equivalent to a BMI program. The numerical solution can then be obtained by exploiting an iterative algorithm. See [18] and references therein for a thorough discussion on related numerical paradigms and available solvers. □

4 A low-demanding Receding Horizon Control Algorithm

This section is devoted to show how the proposed procedure **SOS-V-K**(x) is capable to achieve satisfactory level of control performance within a Receding Horizon Control (RHC) framework. The idea is to resort to a computationally low demanding RHC scheme where most of the computations are carried out off-line. It is generally

recognized that the evaluation of the on-line parts of traditional robust MPC schemes is computationally prohibitive in many practical situations. The problem is especially severe for nonlinear schemes and most of current research on MPC is in fact devoted to reduce such a high computational burden while still ensuring the same level of control performance of the traditional schemes. Examples of these new algorithms include exact [19] and approximate [20] explicit MPC schemes, efficient implementations of MPC via off-line computation of ellipsoidal [15] and polytopic approximations [7] of exact controllable sets. Here, the procedure **SOS-V-K**(x) will be exploited within the Robust RHC scheme proposed in [15] for the uncertain linear time invariant systems.
All the arguments developed in the previous sections allows one to write down a computable RHC scheme, hereafter denoted as **WK-SOS**, which consists of the following algorithm:
Algorithm-WK-SOS
Off-line

0.1 *Given an initial feasible state* x_1*, put* $r = 1$

0.2 *Generate a sequence of control laws* $K_r(\cdot)$*, invariant regions* $\mathcal{E}_r$ *by solving the SOS program* **SOS-V-K**(x_r) *with the additional constraint* $\mathcal{E}_r \subset \mathcal{E}_{r-1}$ *translated as an extra SOS condition in the above* **Lyapunov function synthesis** *phase (3)*

$$-((\alpha - V)s_{16} + (V_{k-1} - 1)) \in \Sigma[x] \text{ with } s_{16} \in \Sigma[x], \partial(V s_{16}) \geq \partial(V_{k-1}) \quad (13)$$

0.3 *Store* $K_r(\cdot)$, $\mathcal{E}_r(\cdot)$ *in a lookup table;*

0.4 *If* $r < N$*, choose a new state* x_{r+1} *s.t.* $x_{r+1} \in \mathcal{E}_r$*, Let* $r = r+1$ *and go to step* 0.2

On-line

1.1 *Given an initial feasible state* $x(0)$ *s.t.* $x(0) \in \mathcal{E}_1$*, put* $t = 0$*;*

1.2 *Given* $x(t)$*, perform a bisection search over the sets* $\mathcal{E}_r$ *in the look-up table to find* $\hat{r} := \arg\min_r$ *s.t.* $x(t) \in \mathcal{E}_r$

1.3 *Feed the plant by the input* $K_{\hat{r}}(x(t))$

1.4 $t \leftarrow t+1$ *and go to step* 1.2

Next lemma ensures that the proposed MPC scheme admits a feasible solution at each time t and the SOS-based input strategy $K_{\hat{r}}(x(t))$ is a stabilizing control law for (1) under (2).

Lemma 1 *Given the system (1), let the off-line steps of proposed scheme have solution at time* $t = 0$*. Then, the on-line part of the* **WK-SOS** *algorithm has solution at each future time instant, satisfies the input constraints and yields an asymptotically stable closed-loop system.*

Proof - It follows by the existence of the sequences $K_r, \mathcal{E}_r$ which ensure that any initial state $x(0) \in \mathcal{E}_1$ can be steered to the origin without constraints violation. In particular, because of the additional constraint $\mathcal{E}_r \subseteq \mathcal{E}_{r-1}$, the regulated state trajectory emanating from the initial state satisfies

$$x(t+1) = \begin{cases} f(x(t)) + g(x(t))K_r(x(t)) & \text{if } x(t) \in \mathcal{E}_r,\ x(t) \notin \mathcal{E}_{r+1},\ r \neq N \\ f(x(t)) + g(x(t))K_N(x(t)) & \text{if } x(t) \in \mathcal{E}_N \end{cases} \quad (14)$$

Then, under both conditions $x(t) \in \mathcal{E}_r$ and $x(t) \notin \mathcal{E}_{r+1}$, $r = 1, \ldots, N-1$, the control law $K_r(\cdot)$ is guaranteed to ultimately drive the state from $\mathcal{E}_r$ into the ellipsoid $\mathcal{E}_{r+1}$ because the Lyapunov difference $V(f(x, \hat{K}_r(x)) - V(x)$ is strictly negative. Finally, the positive invariance of $\mathcal{E}_N$ and the contraction provided by K_N, guarantee that the state remains within $\mathcal{E}_N$ and converges to 0_x. □

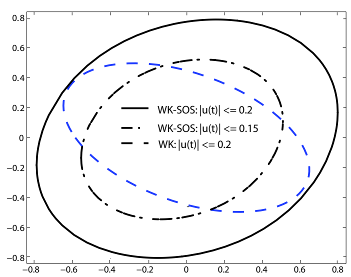
5 Illustrative example

The aim of this section is to give a measure of the improvements achievable by exploiting the SOS programming framework within a RHC framework. To this end, the **Algorithm-WK**, (see [15]) will be contrasted with the proposed **Algorithm-WK-SOS**. The simulations are instrumental to show especially the reduction of conservativeness in terms of achievable basins of attraction, when compared with its linear counterpart (viz. **Algorithm-WK**). All the computations have been carried out on a PC Pentium 4.

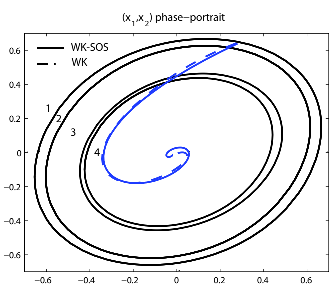
A controlled Van Der Pol nonlinear equation is taken into consideration [21]

$$\begin{cases} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -x_1(t) - (1 - x_1^2(t))x_2(t) + u(t) \end{cases} \tag{15}$$

It has an unstable limit cycle around 0_x, which is a local asymptotically stable equilibrium point. The problem of computing inner approximations of the attraction basin with SOS machinery has been extensively studied (see [21] and references therein). The system (15) has been discretized by using forward Euler differences with a sampling time $T_c = 0.1$ sec. It has been further assumed: weighting matrices $\Psi_x = diag([0.01 \; 0.01])$, and $\Psi_u = 1$ and input saturation constraint $|u(t)| \leq 0.2,\ \forall t$.
As it is well known in literature, the attraction region of the equilibrium point 0_x is enclosed by its limit cycle and we have initially chosen the candidate function $p(x)$ as $p(x) = x_1^2 + x_2^2$. The other design points are here summarized: Candidate Lyapunov function degree: $\partial(V(x)) = 6$; Candidate stabilizing controller degree: $\partial(K(x)) = 4$. The following degrees have been chosen $\partial(s_6) = 2$, $\partial(s_8) = 4$, $\partial(s_9) = 0$, $\partial(s_{11}) = 2$, $\partial(s_{12}) = 2$, $\partial(s_{14}) = 2$, $\partial(s_{16}) = 2$, for the free polynomials in the SOS formulation in order to satisfy the solvability conditions (12). Finally, the quantity ε in (32) has been chosen equal to 10^{-8}. Fig. (a) reports the basins of attraction for the

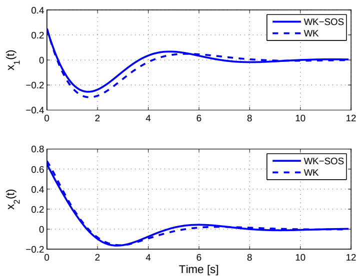


(a) State Attraction Region with input bound constraints - **WK-SOS**, $|u(t)| \leq 0.2$ (Continuous line), $|u(t)| \leq 0.15$ (Dash-dotted line); **WK**, $|u(t)| \leq 0.2$ (Dashed line)

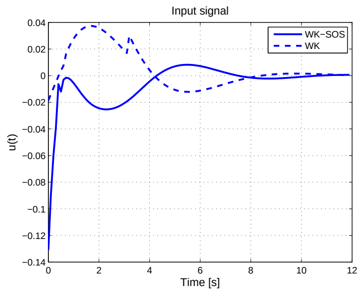


(b) Phase portrait with input bound constraints $|u(t)| \leq 0.2$.

two control schemes. As expected, **WK-SOS** (continuous line) enjoys an enlarged region of feasible initial states w.r.t. **WK**(dashed). Moreover, when one computes the basins of attraction under the more stringent saturation constraint $|u(t)| \leq 0.15$ it results that no solutions exist for the **WK** algorithm whereas a restricted region (dotted in Fig. (a)) is found for **WK-SOS** The results reported in next Figs (b)-(d) have been



(c) State evolutions with input bound constraints $|u(t)| \leq 0.2$



(d) Input signal with input bound constraints $|u(t)| \leq 0.2$

achieved with the input constraint $|u(t)| \leq 0.2$. Four pairs $(K_i, \mathcal{E}_i)$ have been determined with the **SOS-V-K(x)** algorithm initialized with a sequence of four states $x^{set} = \begin{bmatrix} 0.4 & 0.3 & 0.15 & 0.1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The initial state has been set to $x(0) = \begin{bmatrix} 0.25 & 0.68 \end{bmatrix}^T$.

Finally, Figs. (b)-(d) depict respectively: the regulated phase portraits (four invariant regions $\mathcal{E}_i$ are graphically represented), the state regulated response and the control input for the two schemes.

6 Conclusions

In this paper, we have developed an off-line RHC algorithm for constrained polynomial nonlinear systems by means of SOS programming. The advantage of this algorithm is that it provides a set of stabilizing polynomial control laws, corresponding to a nested set of positive invariant regions. Up to our knowledge this is a first attempt in literature to formulate a RHC problem using SOS machinery. Numerical experiments have shown the benefits of the proposed RHC strategy w.r.t. linear embedding MPC schemes and makes SOS based MPC schemes potentially attractive.

References

[1] D. Q. Mayne, J. B. Rawlings., C. V. Rao and P. O. M. Scokaert. Constrained model predictive control: stability and optimality. *Automatica*, 36:789–814, 2000.

[2] D. Angeli, A. Casavola, G. Franzè and E. Mosca. An Ellipsoidal Off-line MPC Scheme for Uncertain Polytopic Discrete-time Systems. *Automatica*, in press, 2008.

[3] F. Allgöwer, T. A. Badgwell, J. S. Qin, J. B. Rawlings, and S. J. Wright. Nonlinear predictive control and moving horizon estimation An introductory overview. *In P. M. Frank, editor, Advances in Control, Highlights of ECC99*, 391–449, Springer Verlag, Berlin, 1999.

[4] H. Chen and F. Allgöwer. A Quasi-Infinite Horizon Nonlinear Model Predictive Control Scheme with Guaranteed Stability. *Automatica*, 34:1205-1217, 1998.

[5] L. Magni, G. De Nicolao, L. Magnani, and R. Scattolini, A stabilizing model-based predictive control for nonlinear systems. *Automatica*, 37:1351-1362, 2001.

[6] C. W. Scherer and C. W. J. Hol. Matrix Sum-of-Squares Relaxations for Robust Semi-Definite Programs. *Mathematical Programmming, Ser. B*, 107:189-211, 2006.

[7] J. Björnberg and M. Diehl. Approximate robust dynamic programming and robustly stable MPC. *Automatica* , 42:777–782, 2006.

[8] G. Chesi. Estimating the domain of attraction for uncertain polynomial systems. *Automatica*, 40:1981-1986, 2004.

[9] P. Parrillo. Structured Semidefinite Programs and Semialgebraic Geometry Methods in Robustness and Optimization. *Ph.D. Thesis*, California Institute of Technology, 2000.

[10] D. Henrion, D. Peaucelle, D. Arzelier and M. Šebek. Ellipsoidal Approximation of the Stability Domain of a Polynomial. *IEEE Transactions on Automatic Control*. 48:2255-2259, 2003.

[11] S. Prajna, P. A. Parrilo and A. Rantzer. Nonlinear control synthesis by convex optimization. *IEEE Transactions on Automatic Control*, 49:310-314, 2004.

[12] I. A. Fotiou, P. Rostalski, P. A. Parrilo and M. Morari. Parametric optimization and optimal control using algebraic geometry methods. *International Journal of Control*, 79:1340-1350, 2006.

[13] T. Raff , C. Ebenbauer , R. Findeisen and F. Allgwer. Nonlinear Model Predictive Control and Sum of Squares Techniques. *Fast Motions in Biomechanics and Robotics, Springer*, 340:325-344, 2006.

[14] Z. Jarvis-Wloszek, R. Feeley, T. Weehong, S. Kunpeng, A. Packard. Some controls applications of sum of squares programming. *Proceedings of the 42nd IEEE Conference on Decision and Control*, Las Vegas, NV, USA, 4676-4681, 2003.

[15] Z. Wan and M. V. Kothare. An efficient off-line formulation of robust model predictive control using linear matrix inequalities. *Automatica*, 39:837-846, 2003.

[16] M. Alamir and G. Bornard. On the stability of receding horizon control of nonlinear discrete-time systems, *System* & *Control Letters*, 23:291-296, 1994.

[17] D. A. Cox, J. Little and D. O'Shea. Using algebraic geometry. Springer Verlag, Berlin, 2004.

[18] J. Löfberg. A Toolbox for Modeling and Optimization in MATLAB. *Proceedings of the (CACSD) Conference.* Taipei, Taiwan, 2004. *Available from* `http://control.ee.ethz.ch/~joloef/yalmip.php`.

[19] A. Bemporad, M. Morari, V. Dua, and E.N. Pistikopoulos. The explicit linear quadratic regulator for constrained systems". *Automatica*, 38:3-20, 2002.

[20] T.A. Johansen. Approximate explicit receding horizon control of constrained nonlinear systems. *Automatica*, 40:293-300, 2004.

[21] W. Tan. Nonlinear Control Analysis and Synthesis using Sum-of-Squares Programming. *Ph.D. Thesis*, California Institute of Technology, 2006, available online at `http://jagger.me.berkeley.edu/papers/thesis.pdf`.

[22] W. Tan and A. Packard. Stability region Analysis using polynomila and composite polynomial Lyapunov functions and Sun-of-Squares programming. *IEEE Transactions on Automatic Control*, 53:565-571, 2008.

[23] B. Ding, Y. Xi, M.T. Cychowski and T. O'Mahony. Improving off-line approach to robust MPC based-on nominal performance cost. *Automatica*, 43:158-163, 2007.

[24] M. Diehl and J. Björnberg. Robust dynamic programming for minmax model predictive control of constrained uncertain systems. *IEEE Transactions on Automatic Control*, 49:2253-2257, 2004.