On the effect of packet acknowledgment on the stability and performance of Networked Control Systems

Emanuele Garone, Bruno Sinopoli and Alessandro Casavola

Abstract This work concerns discrete-time Linear Quadratic Gaussian (LQG) optimal control of a remote plant that communicates with the control unit by means of a packets dropping channel. Namely, the output measurements are sent to the control unit through an unreliable network and the actions decided by the control unit are sent to the plant actuator via the same network. Sensor and control packets may be randomly lost according to a Bernoulli process. In this work we focus on the importance of acknowledgments in the communication between the control unit and the actuators. In the literature two extreme cases have been considered: either guaranteed acknowledgment or complete lack of it. Although very common in practice, the case where the acknowledgment packets can be lost has not been dealt with a sufficient level of detail. In this work we focus on such a case by assuming that also the acknowledgment packets can be lost according to a Bernoulli process. We can show how the partial loss of acknowledgements yields a non classical information pattern [1], making the optimal control law a nonlinear function of the informa-

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Emanuele Garone
Dipartimento di Elettronica, Informatica e Sistemistica, Università degli Studi della Calabria, Via Pietro Bucci, Cubo 42-c, Rende (CS), 87036, Italy, e-mail: egarone@deis.unical.it

Bruno Sinopoli
Department of Electrical and Computer Engineering, Carnegie Mellon University, Pittsburgh, PA 15213, USA e-mail: brunos@ece.cmu.edu

Alessandro Casavola
Dipartimento di Elettronica, Informatica e Sistemistica, Università degli Studi della Calabria, Via Pietro Bucci, Cubo 42-c, Rende (CS), 87036, Italy, e-mail: casavola@deis.unical.it

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tion set. For the special case each observation packet contains the complete state information, we can prove linearity of the optimal controller. Furthermore, we can compute the control law in closed form and show that the stability range increases monotonically with the arrival rate of the acknowledgement packets.

1 Introduction

The ubiquity of networking together with technological advances in embedded sensing, computing and wired/wireless communication is today enabling the use of shared general purpose networks for control system applications, in stark contrast with the traditional use of dedicated connections [2]. Due to inherent unreliability that such a paradigm introduces, the control system community has focused his attention to the implications of using imperfect communication channels to "close the loop". Several aspects (see [3] and [4] for a survey) concerning Networked Control Systems have been analyzed with special attention to limited bandwidth ([5]- [7]), transmission delays ([8]-[12]) and packet dropouts. In this work we will focus on the effect of packet loss on system stability and performance.

Here we consider a generalized formulation of the Linear Quadratic Gaussian (LQG) optimal control problem where sensors and actuators communicate with a remote control unit via a lossy network. Observation and control packet drops can be modeled with independent Bernoulli's processes of parameters $\overline{\gamma}$ and, respectively, $\overline{\nu}$. Please note that the latter description presents two ambiguities which must be solved in order to correctly define the control problem.

First, we need to specify the control signal applied by the actuators in the case that some control packets are lost. The most common approaches presented in literature consist of either providing a zero input [13] when the communication fails or using a zero-holder [14] to maintain the previous applied input. Recently, in [15] it has been shown that, in general, none of the two approaches can be claimed superior to the other. In the rest of the chapter the zero input approach will be used.

The second ambiguity to be addressed is the specification of the information available at each time instant to the control unit. This ambiguity resides in the actuator's communication channel. In fact, because the control unit is the transmitter of the packet, it could either have or not knowledge of the successful command transmission, the latter depending on the presence of an acknowledgment protocol.

Usually in literature the two extreme cases are considered: situations where a perfect acknowledgment is guaranteed at each time instant or where no acknowledgment is provided. Following the framework proposed by Imer [16], approaches related to the first case are usually referred to as TCP-like and UDP-like for the second. Previous work [17, 18, 19] has shown the existence of a critical domain of values for the parameters of the Bernoulli arrival processes, $\overline{\nu}$ and $\overline{\gamma}$, outside of which a transition to instability occurs and the optimal controller fails to stabilize the system. In particular, it has been proven that under TCP-like protocols the crit-

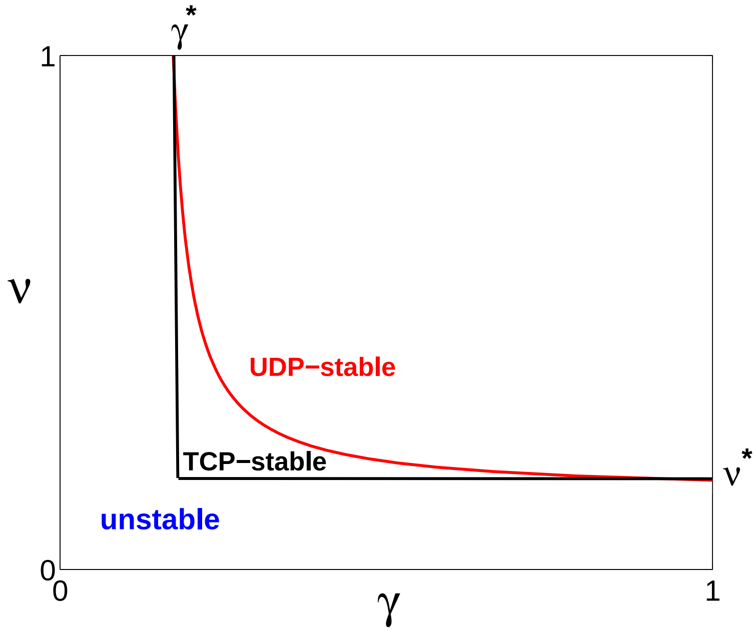

Fig. 1 Stability regions for TCP-like and UDP-like systems (special case with perfect observations). Architecture of the closed-loop system over a communication network. The binary random variables ν_t, γ_t and θ_t indicate whether packets are transmitted successfully.

ical arrival probabilities for the control and observation channels are independent each other. A more involved situation regards UDP-like protocols.

We have also shown that in the TCP-like case the classic separation principle holds, and consequently the controller and estimator can be designed independently. Moreover, the optimal controller is a linear function of the state. In sharp contrast, in the UDP-like case, the optimal controller is in general non-linear. In this case the absence of an acknowledgement structure generates a nonclassical information pattern [1]. Because of the importance of UDP protocols for wireless sensor networks, we have analyzed a special case when the arrival of a sensor packet provides complete knowledge of the state and, despite the lack of acknowledgements, the optimal control design problem yields a linear controller [19]. In this case, the critical arrival probabilities for the control and observation channels are coupled. The stability domain and the performance of the optimal controller degrade considerably as compared with TCP-like protocols as shown in Figure 1.

Also, for the general case, a sub-optimal solution was provided in [20], by designing the optimal linear static regulator, composed by constant gains for both the observer and the controller. This is particularly attractive for sensor networks, where the simplicity of implementation is highly desirable and the complexity issues are a primary concern. Recently Epstein et al. [21] proposed, in the context of UDP-like control, to estimate not only the state of the system, but also a binary variable which indicates wether the previous control packet has been received or not. Such strategy, improves closed-loop performance at the price of a somewhat larger computational complexity.

In this work, similarly to the analysis carried out in [22], we drop the assumption of guaranteed deterministic acknowledgement and assume only that the acknowledgment packets can be lost accordingly to a Bernoulli's process. It is shown that loss of acknowledgement leads once again to a nonclassical information pattern, and we are able to prove that in general the optimal control law is a nonlinear function of the information set. By restricting ourselves to the complete observability case, we are able to solve the LQG problem. We show that probabilistic acknowledgements increase the stability range of the system with respect to UPD-like controls. Furthermore, we can also show that such domains converge to the TCP-like one as the erasure probability for the acknowledgement channel tends to zero.

The remainder of the paper is organized as follows. In Section 2 we provide the problem formulation; in Section 3, we derive the estimator equations; in Section 4, we consider the control problem in the general case; in Section 5 we consider the special case of complete observability; in Section 6 we present an illustrative example and in Section 7 we provide conclusions and directions for future work.

2 Problem and Formulation

Consider the following linear stochastic system with intermittent observation and control packets:

$$\begin{array}{l} x_{k+1} = Ax_k + Bu_k^a + \omega_k \\ u_k^a = \nu_k u_k^c + [1-\nu_k]\, u_k^l \\ y(k) = \gamma_k C x_k + v_k \end{array} \tag{1}$$

where u_k^a is the control input to the actuator, u_k^c is the desired control input computed by the controller, (x_0, ω_k, v_k) are Gaussian, uncorrelated, white, with mean $(\overline{x_0}, 0, 0)$ and covariance (P_0, Q, R) respectively, and γ_k and ν_k are i.i.d. Bernoulli random variable with $P(\gamma_k = 1) = \overline{\gamma_k}$ and $P(\nu_k = 1) = \overline{\nu_k}$. u_k^l is the signal it is locally provided to the actuators in the case $\nu_k = 0$ (the control packet packet to the actuators is lost). In this work we will consider $u_k^l = 0$.

A key point towards the design of any control strategy is the definition of the *Information Set* available to the controller at each time instant. It is usual in literature (see [16]) to consider the following two Information Sets

$$I_k = \begin{cases} F_k = \{\gamma_k y_k, \gamma_k, \nu_{k-1} | k = 0, \ldots, t\} & TCP-like \\ G_k = \{\gamma_k y_k, \gamma_k | k = 0, \ldots, t\} & UDP-like \end{cases}$$

As depicted in Figure 2, the difference between the two Information Sets is the availability of the control packet acknowledgement i.e. ν_{k-1}. While for the "TCP-like" case, several useful and important results (separation principles, linear quadratic gaussian optimal control, etc...) are known, it is well known from networks and

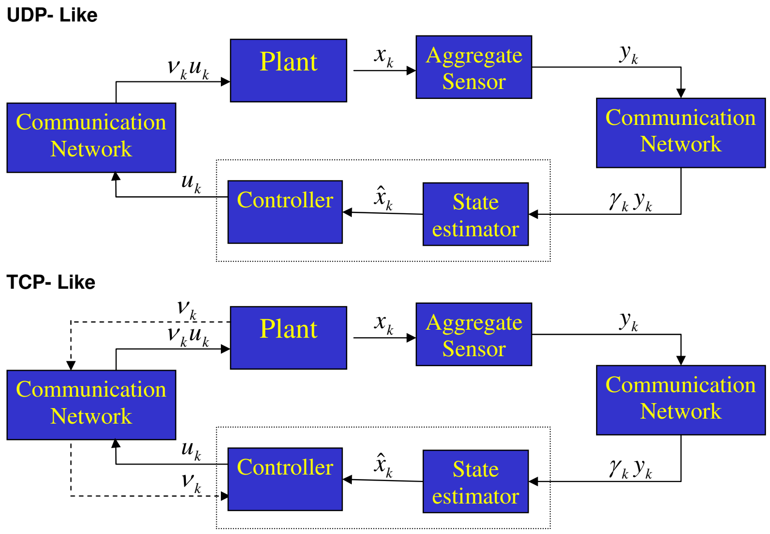

Fig. 2 TCP-like and UDP-like Framework: Bernoulli variables γ_k, ν_k model the packets arrival through the network. Dashed lines represent the communication from the actuators to the control unit. Note that in the UDP case no acknowledgment traffic exists. On the contrary, in the TCP case instead, all the acknowledgment packets sent from the actuators are supposed to arrive at the control unit.

computer science literature that guaranteeing a deterministic ”perfect” acknowledgements is in general a very difficult task and, in the case the acknowledgement packets use unreliable channels, theoretically impossible as it constitutes a particular case of the *two-armies problem* (see [23]).

On the other hand it is extremely difficult [19] to design optimal estimators and controllers under the information set G_k, since the separation principle does not hold and it can be shown that the optimal control is not linear. Moreover, performance and stability regions can be highly affected, due to the fact that no ”real” information on the actual input is exploited.

In many practical cases, it is reasonable to use communication channels where acknowledgements are provided although they can be dropped with a probability which depends on both the channel reliability and the protocol employed. This means that, during each process, we have a non-zero probability to lose the acknowledgment packet. In order to formalize this assumption, let us denote by θ_k the Bernoulli variable which models the acknowledgment arrivals and by $\overline{\theta}$ its arrival probability. The new Information Set can then be defined as follows

$$E_k = \{\gamma_k y_k, \gamma_k, \theta_{k-1}, \theta_{k-1}\nu_{k-1} | k = 0, \dots, t\} \tag{2}$$

and the overall closed loop system is depicted in Figure 3.

Let us now define $u^{N-1} = \{u_0, u_1, \dots, u_{N-1}\}$ as the set of all the input values between time instants 0 and $N-1$. In this work we will consider the LQG con-

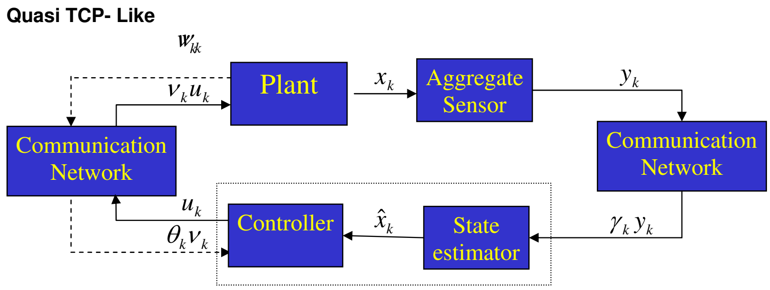

Fig. 3 Quasi TCP-like Framework: Dashed lines represent the communication from the actuators to the control unit. Note that in this case the acknowledgment packets sent from the actuators can be dropped according to the random Bernoulli variable θ_k.

trol problem, i.e. we will look for a control input sequence u^{N-1*}, function of the information set E_k, that solves the following optimization problem

$$J_N^*(\overline{x}_0, P_0) = min_{u_k=g_k(E_k)} J_N(u^{N-1}, \overline{x}_0, P_0), \tag{3}$$

where the cost function $J_N\left(u^{N-1}, \overline{x}_0, P_0\right)$ is defined as follows:

$$J_N\left(u^{N-1}, \overline{x}_0, P_0\right) = E\left[x_N^T W_N x_N + \sum_{k=0}^{N-1} x_k^T W_k x_k + u_k^{aT} U_k u_k^a \middle| u^{N-1}, \overline{x}_0, P_0\right]. \tag{4}$$

3 Estimator design

By the knowledge of the information set (2), the one-step prediction can be easily written as follows:

$$\hat{x}_{k+1|k} = A\hat{x}_{k|k} + \theta_k \nu_k B u_k + (1-\theta_k)\overline{\nu} B u_k. \tag{5}$$

Using (5), it is possible to rewrite the prediction error as follows:

$$\begin{aligned} e_{k+1|k} &= x_{k+1} - \hat{x}_{k+1|k} = \\ &= Ax_k + \nu_k B u_k + \omega_k - A\hat{x}_{k|k} + \theta_k \nu_k B u_k - (1-\theta_k)\overline{\nu} B u_k = \\ &= Ae_{k|k} + (\nu_k - \theta_k \nu_k - (1-\theta_k)\overline{\nu}) B u_k + \omega_k. \end{aligned} \tag{6}$$

We can then compute the associated covariance of the one-step prediction error:

$$\begin{aligned} P_{k+1|k} &= E\left[e_{k+1|k} e_{k+1|k}^T | E_k, \theta_k, \theta_k \nu_k\right] = \\ &= E\left[Ae_{k|k} e_{k|k}^T A | E_k\right] + E\left[\omega_k \omega_k^T | E_k\right] + \\ &\quad + E\left[(\nu_k - \theta_k \nu_k - (1-\theta_k)\overline{\nu})^2 | E_k, \theta_k, \theta_k \nu_k\right] B u_k u_k^T B^T, \end{aligned}$$

obtaining

$$P_{k+1|k} = AP_{k|k}A^T + Q + (1-\theta_k)(1-\overline{\nu})\overline{\nu}\left[Bu_k u_k^T B^T\right]. \tag{7}$$

Equations (5), (6) and (7) represent the predictions of the Kalman Filter for the system (1). The correction steps, instead, are the classical ones reported in ([32]):

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + \gamma_{k+1}K_{k+1}\left(y_{k+1} - Cx_{k+1|k}\right) \tag{8}$$
$$P_{k+1|k+1} = P_{k+1|k} - \gamma_{k+1}K_{k+1}CP_{k+1|k} \tag{9}$$
$$K_{k+1} = P_{k+1|k}C^T\left(CP_{k+1|k}C^T + R\right)^{-1} \tag{10}$$

Remark 1. Note that:

$$\begin{aligned}
&\theta_k = 1 \Rightarrow P_{k+1|k} = AP_{k|k}A + Q\\
&\theta_k = 0 \Rightarrow P_{k+1|k} = AP_{k|k}A + Q + \overline{\nu}(1-\overline{\nu})\left[Bu_k u_k^T B^T\right].
\end{aligned}$$

This implies that, at each time k, the prediction switches between the ”TCP-like” predictions and the ”UDP-like” ones, depending on the instantaneous value of θ_k.

4 Optimal Control - general case

In this section we will show that, in the presence of stochastic acknowledgements, the optimal control law is not a linear function of the state and that the estimation and control designs cannot be treated separately. In order to prove such a claim, it is sufficient to consider the following simple counterexample.

Example 1 - Consider a simple scalar discrete-time Linear Time-Invariant (LTI) system with a single sensor and a single actuator, i.e. A=B=C=W$_N$=W$_k$=R=1,U$_k$=Q=0. We can define

$$\begin{aligned}
V(N) &= E\left[x_N^T W_N x_N | E_N\right]\\
&= E\left[x_N^2 | E_N\right].
\end{aligned}$$

For $k = N-1$ we have:

$$\begin{aligned}
V_{N-1}(x_{N-1}) &= \min_{u_N} E\left[x_{N-1}^2 + V_N(x_N)|E_{N-1}\right] =\\
&= \min_{u_N} E\left[x_{N-1}^2 + x_N^2|E_{N-1}\right] =\\
&= \min_{u_N} E\left[x_{N-1}^2 + (x_{N-1} + \nu_{N-1}u_{N-1})^2|E_{N-1}\right],
\end{aligned} \tag{11}$$

and then finally

$$V_{N-1}(x_{N-1}) = E\left[2x^2_{N-1|N-1}|E_{N-1}\right] + \min_{u_N} \overline{\nu}\left(u^2_{N-1} + 2\hat{x}_{N-1|N-1}u_{N-1}\right). \tag{12}$$

If we differentiate the latter, we obtain the following optimal input :

$$u^*_{N-1} = -\hat{x}_{N-1|N-\hat{1}} \tag{13}$$

If we substitute (13) in (11) the cost becomes:

$$\begin{aligned} V_{N-1}(x) &= E\left[2x^2_{N-1}|E_{N-1}\right] - \overline{\nu}\,\hat{x}^2_{N-1|N-1} = \\ &= (2-\overline{\nu})E\left[x^2_{N-1}|E_{N-1}\right] - \overline{\nu}P_{N-1|N-1}. \end{aligned} \tag{14}$$

Let us focus now on the covariance matrix:

$$\begin{aligned} P_{N-1|N-1} &= P_{N-1|N-2} - \gamma_{N-1}\frac{P^2_{N-1|N-2}}{(P_{N-1|N-2}+1)} = \\ &= P_{N-1|N-2} - \gamma_{N-1}\left(P_{N-1|N-2} - 1 + \frac{1}{(P_{N-1|N-2}+1)}\right) \end{aligned} \tag{15}$$

Because

$$P_{N-1|N-2} = P_{N-2|N-2} + (1-\theta_{N-2})(1-\overline{\nu})\overline{\nu}u^2_{N-2} \tag{16}$$

one finds that

$$\begin{aligned} &E\left[P_{N-1|N-1}|E_{N-2}\right] = P_{N-2|N-2} + \left(1-\overline{\theta}\right)(1-\overline{\nu})\overline{\nu}u^2_{N-2} + \\ &-\overline{\gamma}\Big(P_{N-2|N-2} + \left(1-\overline{\theta}\right)(1-\overline{\nu})\overline{\nu}\,u^2_{N-2} - 1 + \overline{\theta}\tfrac{1}{P_{N-2|N-2}} + \\ &\left(1-\overline{\theta}\right)\tfrac{1}{P_{N-2|N-2}+(1-\overline{\nu})\overline{\nu}u^2_{N-2}}\Big). \end{aligned} \tag{17}$$

Finally, we get

$$\begin{aligned} V_{N-2}(x) &= \min_{u_{N-2}} E\left[x^2_{N-2} + V_{N-1}(x_{N-1})|E_{N-2}\right] = \\ &= (3-\overline{\nu})E\left[x^2_{N-1}|E_{N-2}\right] + \min_{u_{N-2}} P_{N-2|N-2} + \\ &\quad + \left(1-\overline{\theta}\right)(1-\overline{\nu})\overline{\nu}u^2_{N-2} - \overline{\gamma}\Big(P_{N-2|N-2} + \\ &\quad + \left(1-\overline{\theta}\right)(1-\overline{\nu})\overline{\nu}u^2_{N-2} - 1 + \overline{\theta}\tfrac{1}{P_{N-2|N-2}} + \\ &\quad + \left(1-\overline{\theta}\right)\tfrac{1}{P_{N-2|N-2}+(1-\overline{\nu})\overline{\nu}u^2_{N-2}}\Big) \end{aligned} \tag{18}$$

The first terms within the last parenthesis in (18) are convex quadratic functions of the control input u_{N-2}, however the last term is not. Then, the optimal control law is, in general, a nonlinear function of the information set E_k. □

There are only two cases where the optimal control is linear. The first case is when $\overline{\theta} = 1$ (TCP-Case). This corresponds to the TCP-like case studied in [32]. The second case is when the measurement noise covariance is zero ($R = 0$) and any delivered packet contains full state information, i.e. $Rank(C) = n$. In fact, this would mean that, at each time instant k, $\gamma_k C$ is either zero or full column-rank. If it is zero, the dependence is linear. If $\gamma_k C$ is full column-rank, it is equivalent to having an exact measurement of the actual state (see [32]). However, it is important to remark that in such a second case the separation principle still does not hold, since the control input affects the estimator error covariance. These results can be summarized in the following Theorem.

Theorem 1. *Let us consider the stochastic system defined in (1) with horizon* $N \geq 2$. *Then:*

- *if* $\overline{\theta} < 1$, *the separation principle does not hold*
- *The optimal control feedback* $u_k = f_k^*(E_k)$ *that minimizes the cost functional defined in Equation (4) is, in general, a nonlinear function of information set* E_k
- *The optimal control feedback* $u_k = f_k^*(E_k)$ *is a linear function of the estimated state if and only if one of the following conditions holds true:*
  - $\overline{\theta} = 1$
  - $Rank(C) = n$ *and* $R = 0$

□

In the next Section we will focus on the case $Rank(C) = n$, and $R = 0$. In particular, we will compute the optimal control and we will show that, in the infinite horizon scenario, the optimal state-feedback gain is constant, i.e. $L_k^* = L^*$ and can be computed as the solution of a convex optimization problem.

5 Optimal Control – Rank(C)=n, R=0 case

Without loss of generality we can assume $C = I$. Because of the hypothesis of no measurement noise, i.e. $R = 0$, it is possible to simply measure the state x_k when a packet is delivered. The estimator equations then simplify in the following way:

$$
\begin{aligned}
K_{k+1} &= I && (19)\\
P_{k+1|k} &= AP_{k|k}A + Q + (1-\theta_k)(1-\overline{\nu})\overline{\nu}\left[Bu_k u_k^T B^T\right] && (20)\\
P_{k+1|k+1} &= (1-\gamma_{k+1})P_{k+1|k} && (21)\\
&= (1-\gamma_{k+1})\left(AP_{k|k}A + Q + \right.\\
&\quad \left. + (1-\theta_k)(1-\overline{\nu})\overline{\nu}\left[Bu_k u_k^T B^T\right]\right)\\
E\left[P_{k+1|k+1}|E_k\right] &= (1-\overline{\gamma})\left(AP_{k|k}A + Q\right. && (22)\\
&\quad \left. + \left(1-\overline{\theta}\right)(1-\overline{\nu})\overline{\nu}\left[Bu_k u_k^T B^T\right]\right).
\end{aligned}
$$

In the last equation the independence of $E_k, \gamma_{k+1}, \theta_k$ is exploited. By following the classical dynamic programming approach to solve optimal control problems, it can be seen that the value function $V_k^*(x_k)$ can be written as follows:

$$\begin{aligned} V_k(x_k) &= \hat{x}_{k|k}^T S_k \hat{x}_{k|k} + trace\left(T_k P_{k|k}\right) + trace\left(D_k Q\right) = \\ &= E\left[x_{k|k}^T S_k x_{k|k}\right] + trace\left(H_k P_{k|k}\right) + trace\left(D_k Q\right) \end{aligned} \tag{23}$$

for each $k = N, \ldots, 0$ where $H_k = T_k - S_k$. This is clearly true for $k = N$. In fact, we have

$$\begin{aligned} V_N(x_N) &= E\left[x_N^T W_N x_N | E_N\right] \\ &= \hat{x}_{N|N}^T W_N \hat{x}_{N|N} + trace\left(W_N P_{N|N}\right), \end{aligned}$$

and the statement is satisfied by $S_N = T_N = W_N, D_N = 0$. Let us suppose that (23) holds true for $k+1$ and let us show by induction that it holds true for k as well

$$\begin{aligned} V_k(x_k) =& \min_{u_k} E\left[x_k^T W_k x_k + \nu_k u_k^T U_k u_k + V_{k+1}(x_{k+1}) | E_k\right] = \\ & \min_{u_k} E\left[x_k^T W_k x_k | E_k\right] + \overline{\nu} u_k^T U_k u_k + E\left[x_{k+1}^T S_{k+1} x_{k+1} | E_k\right] + \\ & trace\left(H_{k+1} P_{k+1|k+1}\right) + trace\left(D_{k+1} Q\right) = \\ =& \min_{u_k} E\left[x_k^T W_k x_k | E_k\right] + \overline{\nu}_k u_k^T U_k u_k + trace\left(D_{k+1} Q\right) + \\ & + trace\left(H_{k+1}\left((1-\overline{\gamma})\left(A P_{k|k} A + Q + (1-\theta_k)\,\overline{\nu}\,(1-\overline{\nu})\left[B u_k u_k^T B^T\right]\right)\right)\right) \\ & + E\Bigg[\left(A x_{k|k} + \theta_k \nu_k B u_k + (1-\theta_k)\,\overline{\nu} B u_k\right)^T S_{k+1} \\ & \left(A x_{k|k} + \theta_k \nu_k B u_k + (1-\theta_k)\,\overline{\nu} B u_k\right) \Bigg| E_k\Bigg]. \end{aligned}$$

Further manipulation yields:

$$\begin{aligned} V_k(x_k) =& \min_{u_k} E\Big[x_k^T W_k x_k + \overline{\nu}_k u_k^T U_k u_k + \left(x_{k|k}^T A^T S_{k+1} A x_{k|k}\right) + \\ & + \left(\theta_k \nu_k u_k^T B^T B u_k\right) + \left((1-\theta_k)\,\overline{\nu} u_k^T B^T B u_k^T\right) + 2\theta_k \nu_k x_{k|k}^T A^T S_{k+1} B u_k + \\ & + 2(1-\theta_k)\,\overline{\nu} x_{k|k}^T A^T S_{k+1} B u_k | E_k\Big] + trace\left(D_{k+1} Q\right) + \\ & + trace\left(H_{k+1}\left((1-\overline{\gamma})\left(A P_{k|k} A + Q + \left(1-\overline{\theta}\right)\overline{\nu}\,(1-\overline{\nu})\left[B u_k u_k^T B^T\right]\right)\right)\right) = \\ =& E\left[x_{k|k}^T\left(W_k + A^T S_{k+1} A\right) x_{k|k}\right] + (1-\overline{\gamma})\, trace\left(H_{k+1}\left(\left(A P_{k|k} A + Q\right)\right)\right) + \\ & + \min_{u_k} \overline{\nu}\left(u_k^T\left(U_k + B^T\left(S_{k+1} + \left(1-\overline{\theta}\right)(1-\overline{\nu})\,\overline{\nu} H_{k+1}\right) B\right) u_k^T\right) + \\ & + 2\overline{\nu}\left(x_{k|k}^T A^T S_{k+1} B u_k\right) + trace\left(D_{k+1} Q\right). \end{aligned}$$

Since $V_k(x_k)$ is a convex quadratic function w.r.t. u_k, the minimizer is the solution of $\partial V_k(x_k)/\partial u_k = 0$, given by:

$$u_k^* = -\left(U_k + B^T\left(S_{k+1} + \overline{\alpha} H_{k+1}\right)B\right)^{-1}\left(B^T S_{k+1} A x_{k|k}\right) = L_k x_{k|k}, \tag{24}$$

where $\overline{\alpha} = (1-\overline{\gamma})\left(1-\overline{\theta}\right)(1-\overline{\nu})\overline{\nu}$. The optimal control is a linear function of the estimated state $x_{k|k}$. Substituting back (24) into the value function we get:

$$\begin{aligned} V_k(x_k) = {} & trace\left((1-\overline{\gamma})H_{k+1}\left(\left(AP_{k|k}A\right)\right)\right) + trace\left(\left((1-\overline{\gamma})T_{k+1} + D_{k+1}\right)Q\right) + \\ & + E\left[x_{k|k}^T\left(W_k + A^T S_{k+1}A\right)x_{k|k}\right] - \overline{\nu} x_{k|k}^T\left(A^T S_{k+1} B L_k\right)x_{k|k}, \end{aligned}$$

which becomes

$$\begin{aligned} V_k(x_k) = {} & trace\left((1-\overline{\gamma})H_{k+1}\left(\left(AP_{k|k}A\right)\right)\right) + \\ & + E\left[x_{k|k}^T\left(W_k + A^T S_{k+1}A\right)x_{k|k} + \left(\overline{\nu}\left(x_{k|k}^T A^T S_{k+1}B\right)L_k x_{k|k}\right)\right] + \\ & + trace\left(\left(D_{k+1} + (1-\overline{\gamma})T_{k+1}\right)Q\right) - trace\left(\left(\overline{\nu}A^T S_{k+1} B L_k P_{k|k}\right)\right). \end{aligned}$$

Finally, we obtain

$$\begin{aligned} V_k(x_k) = {} & trace\left(\left(D_{k+1} + (1-\overline{\gamma})H_{k+1}\right)Q\right) + \\ & + E\left[x_{k|k}^T\left(W_k + A^T S_{k+1}A\right)x_{k|k} + \left(\overline{\nu}\left(x_{k|k}^T A^T S_{k+1}B\right)L_k x_{k|k}\right)\right] + \\ & + trace\left(\left(\left((1-\overline{\gamma})A^T H_{k+1}A - \overline{\nu}A^T S_{k+1}BL_k\right)P_{k|k}\right)\right). \end{aligned}$$

From the last equation we see that the value function can be written as in (23) if and only if the following equations are satisfied:

$$S_k = W_k + A^T S_{k+1} A + \overline{\nu}\left(A^T S_{k+1} B\right) L_k \tag{25}$$

$$T_k = (1-\overline{\gamma}) A^T T_{k+1} A + W_k + \overline{\gamma} A^T S_{k+1} A \tag{26}$$

$$D_k = D_{k+1} + (1-\overline{\gamma})\, T_{k+1} + \overline{\gamma} S_{k+1}. \tag{27}$$

Remark 2. Notice that, if $\overline{\theta} \to 0$, the control design system soon regresses to the UDP-like case.

The optimal minimal cost for the finite horizon, $J_N^* = V_0(x_0)$ is then given by:

$$J_N^* = \bar{x_0}^T S_0 x_0 + trace(S_0 P_0) + trace(D_0 Q).$$

For the infinite horizon optimal controller, necessary and sufficient conditions for the average minimal cost $J_\infty^* = \lim\limits_{N\to\infty} \frac{1}{N} J_N^*$ to be finite, are that the coupled recurrent equations (26) and (25) converge to a finite value S_∞ and T_∞ as $N \to \infty$.

Theorem 2. *Consider system (1) and the problem of minimizing the cost function (4) within the class of admissible policies $u_k = f(E_k)$. Assume also that $R = 0$ and $rankC = n$. Then:*

1. *The optimal estimator gain is constant and in particular $K_k = I$ if $C = I$.*
2. *The infinite horizon optimal control exists if and only if there exist positive definite matrices $S_\infty, T_\infty > 0$ such that $S_\infty = \Phi_S(S_\infty, T_\infty)$ and $T_\infty = \Phi_T(S_\infty, T_\infty)$, where Φ_S and Φ_T are:*

$$\begin{aligned}\Phi_S(S_k, W_k) &= W_k + A^T S_k A - \overline{\nu}\left(A^T S_k B\right) \\ &\quad \left(U_k + B^T((1-\overline{\alpha})S_{k+1} + \overline{\alpha}T_{k+1})B\right)^{-1}\left(B^T S_{k+1} A\right) \qquad (28)\\ \Phi_T(S_k, T_k) &= (1-\overline{\gamma})A^T T_{k+1} A + W_k + \overline{\gamma} A^T S_{k+1} A. \qquad (29)\end{aligned}$$

3. *The infinite horizon optimal controller gain is constant: $\lim_{k\to\infty} L_k = L_\infty$*

$$L_\infty = -\left(U + B^T((1-\overline{\alpha})S_\infty + \overline{\alpha}T_\infty)B\right)^{-1}\left(B^T S_\infty A\right). \qquad (30)$$

4. *A necessary condition for existence of $S_\infty, T_\infty > 0$ is*

$$\begin{aligned}&1 - |A|^2\left(1 - \frac{\overline{\nu}}{(1-\overline{\alpha}) + \overline{\alpha}\frac{\overline{\gamma}|A|^2}{1-(1-\overline{\gamma})|A|^2}}\right) \geq 0\\ &\overline{\gamma} > 1 - \tfrac{1}{|A|^2},\end{aligned} \qquad (31)$$

   where $|A| = \max_i |\lambda_i(A)|$ is the largest eigenvalue of the matrix A. This condition is also sufficient if B is square and invertible.
5. *The expected minimum cost for the infinite horizon scenario converges to:*

$$J_\infty^* = \lim_{k\to\infty} \frac{1}{N} J_N^* = trace\left(((1-\overline{\gamma})T_k + \overline{\gamma}S_k)Q\right). \qquad (32)$$

Proof. Proof is omitted for space limitations. Please refer to [33] for a complete proof.

6 Example 2 - The Batch Reactor

The goal of this example is to show how stability and control performance are affected by the acknowledgment packet arrival probability. Let us consider an unstable batch reactor (see [34], pp. 62) with full sensing. Due to physical distance between sensors and actuators it may be convenient to connect sensors and actuators through an (eventually wireless) channel. The linearized process model is:

$$\dot{x}(t) = \begin{pmatrix} 1.3800 & -0.2077 & 6.715 & -5.676 \\ -0.5814 & -4.29 & 0 & 0.675 \\ 1.067 & 4.273 & -6.654 & 5.893 \\ 0.048 & 4.273 & 1.343 & -2.104 \end{pmatrix} x(t) + \begin{pmatrix} 0 & 0 \\ 5.679 & 0 \\ 1.136 & -3.146 \\ 1.136 & 0 \end{pmatrix} u(t)$$

$$y(t) = x(t).$$

Such a model is discretized with a sampling time $\tau = 0.1s$ and embedded in our networked scheme:

$$x_{k+1} = \begin{pmatrix} 1.1782 & 0.0015 & 0.5116 & -0.4033 \\ -0.0515 & 0.6619 & -0.0110 & 0.0613 \\ 0.0762 & 0.3351 & 0.5607 & 0.3824 \\ -0.0006 & 0.3353 & 0.0893 & 0.8494 \end{pmatrix} x_k + \begin{pmatrix} 0.0045 & -0.0876 \\ 9.4672 & 0.0012 \\ 0.2132 & -0.2353 \\ 0.2131 & -0.0161 \end{pmatrix} \nu_k u_k$$

$$y_k = \gamma_k x_k.$$

Figure 4 shows the different necessary stability regions with respect to $\overline{\nu}$ and $\overline{\gamma}$, parameterized by the acknowledgement probability $\overline{\theta}$. In particular, it is possible to show that, as $\overline{\theta} \to 1$ the stability region converges to the one computed for the TCP-like case. Figure 5 shows the expected cost J_N^* as a function of $\overline{\theta}$ for $\overline{\gamma} = 0.5, Q = I$, $U = I, W = I$. It is worth to notice that the influence of packet acknowledgment on performance decreases as $\overline{\nu}$ tends to 1. This matches with the intuition that in case of perfect communication between the control unit and the actuators (i.e. $\overline{\nu} = 1$) the acknowledgment does not carry any additional information.

7 Conclusions

In this work we analyzed a generalized version of the LQG control problem in the case where both observation and control packets may be lost during transmission over a communication channel. This situation arises frequently in distributed systems where sensors, controllers and actuators reside in different physical locations and have to rely on data networks to exchange information. In this context controller design heavily depends on the communication protocol used. In fact, in TCP-like protocols, acknowledgements of successful transmissions of control packets are provided to the controller, while in UDP-like protocols, no such feedback is provided. In the first case, the separation principle holds and the optimal control is a linear function of the state. As a consequence, controller and estimator design problems are decoupled. UDP-like protocols present a much more complex problem. We have shown that the even partial lack of acknowledgement of control packets results in the failure of the separation principle. Estimation and control are now intimately coupled. In addition, the LQG optimal control is, in general, nonlinear in the estimated state. In the particular case where the observation packet contains full state information the optimal controller is linear in the state. In this particular case we could show how the partial presence of acknowledgement increases both the perfor-

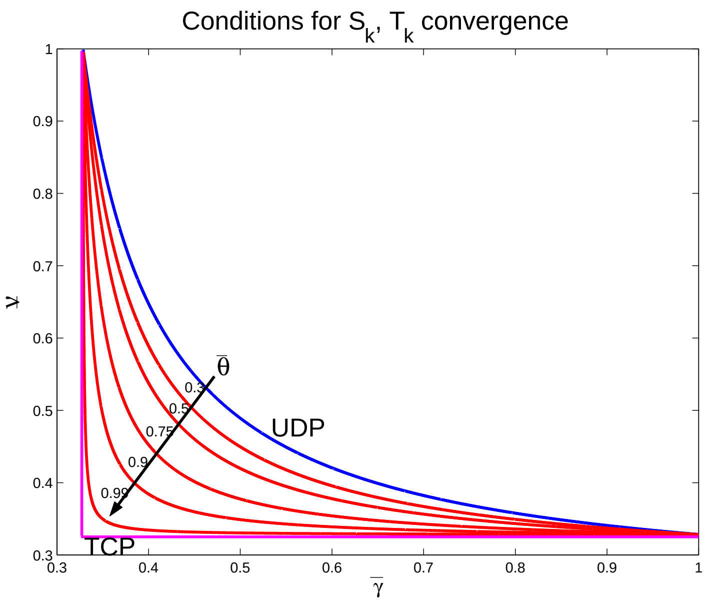

Fig. 4 Region of convergence relative to measurement packet arrival probability $\bar{\gamma}$, and the control packet arrival probability $\bar{\nu}$, parameterized into the acknowledgment packet arrival probability $\bar{\theta}$. Blue and magenta lines depict respectively the bound of the UDP-like and TCP-like cases.

mance and the stability range of the overall system, which converges to the TCP-like with deterministic acknowledgements as the arrival rate for the acknowledgement packets tends to one.

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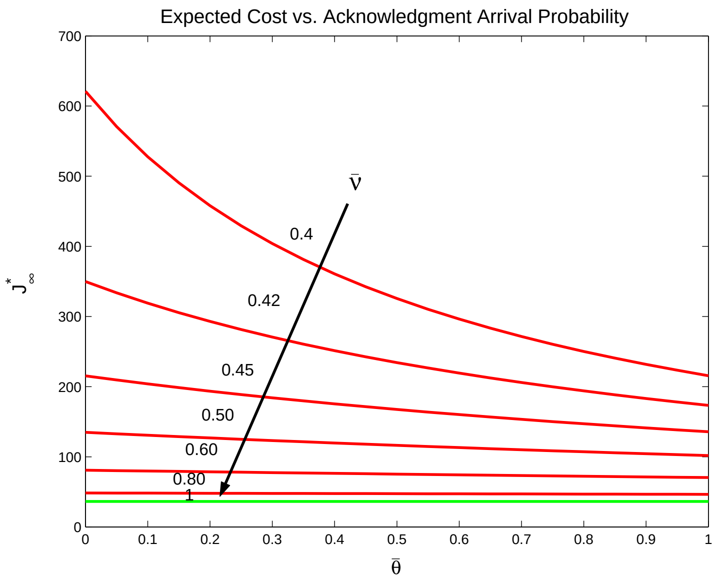

Fig. 5 Expected cost J_N^* as a function of the acknowledgment packet arrival $\overline{\theta}$ for a fixed measurement packet arrival $\overline{\gamma} = 0.5$ and for different control packet arrival $\overline{\nu}$. Green line denotes perfect communication between control unit and actuator.

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