Automatica 47 (2011) 1294–1303

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Sensorless supervision of linear dynamical systems: The Feed-Forward Command Governor approach☆

Emanuele Garone [a,*], Francesco Tedesco [b], Alessandro Casavola [b]

[a] Université Libre de Bruxelles, 50 Av. F.D. Roosvelt, B-1050 Brussels, Belgium

[b] Università degli Studi della Calabria, Via Pietro Bucci, Cubo 42-c, Rende (CS), 87036, Italy

ARTICLE INFO

Article history:
Received 19 January 2010
Received in revised form
14 June 2010
Accepted 28 October 2010
Available online 29 April 2011

Keywords:
Control of constrained systems
Reference management

ABSTRACT

This paper proposes a novel class of Command Governor (CG) strategies for input and state-related constrained discrete-time LTI systems subject to bounded disturbances in the absence of explicit state or output measurements. While in traditional CG schemes the set-point manipulation is undertaken on the basis of either the actual measure of the state or its suitable estimation, it is shown here that the CG design problem can be solved, with limited performance degradation and with similar properties, also in the case that such an explicit measure is not available. This approach, which will be referred to as the *Feed-Forward CG* (FF-CG) approach, may be a convenient alternative CG solution in all situations whereby the cost of measuring the state may be a severe limitation, e.g. in distributed or decentralized applications. In order to evaluate the method proposed here, numerical simulations on a physical example have been undertaken and comparisons with the standard state-based CG solution reported.

© 2011 Elsevier Ltd. All rights reserved.

1. Introduction

The Command Governor (CG) is a nonlinear device which is added to a primal compensated plant that, in the absence of it, is designed so as to exhibit stability and good tracking performance. The main objective of the CG is to modify, whenever necessary, the prescribed reference signal to be supplied to such a pre-compensated system if the unmodified application would lead to constraint violations. Many mature assessments of the CG approach can be found in Gilbert and Kolmanovsky (1999), Gilbert and Kolmanovsky (2001), Gilbert, Kolmanovsky, and Tin Tan (1995) and Gilbert and Tin-Tan (1991). In particular, CG schemes dealing with disturbances were considered in Casavola, Mosca, and Angeli (2000) and Gilbert and Kolmanovsky (1999), with model uncertainties in Bemporad and Mosca (1998) and Casavola et al. (2000) and with partial state information in Angeli, Casavola, and Mosca (2001). For specific results on CG applied to nonlinear systems see e.g. Angeli, Casavola, and Mosca (1999), Angeli and Mosca (1999), Bemporad (1998) and Gilbert and Kolmanovsky (2001), for networked master/slave frameworks Casavola, Papini, and Franzè (2006) and for recent results on hybrid Piecewise-Affine systems Falcone, Borrelli, Peckar, and Steward (2009). Different perspectives on reference management strategies have been reported in Hatanaka and Takaba (2005) and Sugie and Yamamoto (2001). For recent applications see Cirre, Berenguel, Valenzuela, and Klempous (2009) and Mattei, Famularo, and Labate (2009). In the above traditional contexts, the CG action is determined on the basis of the knowledge of the actual measured state. In this paper, on the contrary, we will introduce a novel solution to the CG problem, hereafter referred to as the *Feed-Forward CG* (FF–CG) approach, that, at the price of some additional conservativeness, is able to accomplish the CG task in the absence of an explicit measure of the state. The idea behind such an approach is that, if sufficiently slow and smooth transitions in the reference modifications are acted by the CG unit, the state evolutions remain *not too far* from the space of feasible steady-states because of the asymptotic stability of the system at hands.

The peculiarities of the proposed FF–CG scheme are of interest in all applications whereby the measure or the estimation of the state is problematic. It is expected to be an attractive solution especially for decentralized frameworks because it would alleviate the need to make the entire aggregate state (or part of it) known to all agents at each time instant, the latter being unrealistic or requiring unrealistic communication infrastructures in some large-scale applications. A preliminary version of this strategy has been proposed in Garone, Tedesco, and Casavola (2009) where

☆ This paper was supported by the Belgian Network DYSCO (Dynamical Systems, Control and Optimization) funded by the Interuniversity Attraction Poles Program, initiated by the Belgian State, Science Policy Office. This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor Masayuki Fujita under the direction of Editor Ian R. Petersen. The scientific responsibility rests with its authors.

* Corresponding author. Tel.: +39 349 6150308; fax: +32 0 2 6502677.
E-mail addresses: egarone@ulb.ac.be (E. Garone), ftedesco@deis.unical.it (F. Tedesco), casavola@deis.unical.it (A. Casavola).

0005-1098/$ – see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.automatica.2011.01.034

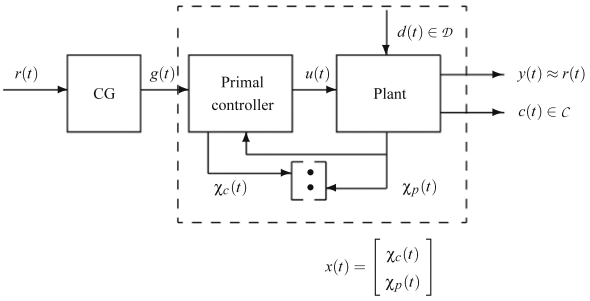

Fig. 1. Feedforward CG structure (without state measurement).

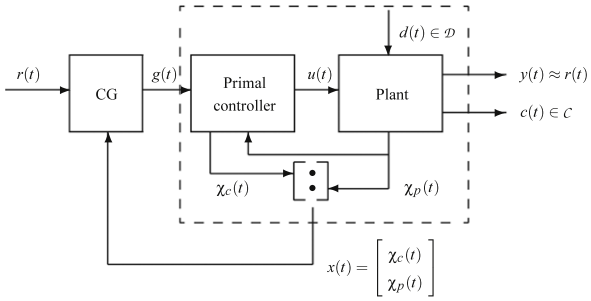

Fig. 2. Classical CG structure (based on state measurement).

it was instrumental for the design of distributed CG strategies for large-scale networked applications. Here, the analysis is extended further on by considering the presence of bounded disturbances that might deteriorate the tracking performance. Moreover, a tighter characterization of the feasible sets underlying the CG action computation is also presented here. This allows the achievement of improved tracking performance with respect to the solution proposed in Garone et al. (2009).

The paper is organized as follows. In Section 2, the CG problem is defined and some preliminary aspects on the idea of the FF–CG algorithm are discussed. In Section 3, the FF–CG scheme is carefully introduced and its properties investigated. For clarity of exposition some of the technical details underlying the derivation of the FF–CG scheme are postponed in Section 4. Finally, in Section 5 numerical simulations are reported in order to show the effectiveness of the proposed algorithm and the comparisons with standard CG solutions. Some conclusions end the paper.

2. System description and problem formulation

A schematic of the CG control structure of interest is depicted in Fig. 1. Let the closed-loop system, consisting of the plant and primal controller of Fig. 1, be described by the following discrete-time model

$$\begin{cases} x(t+1) = \Phi x(t) + Gg(t) + G_d d(t) \\ y(t) = H_y x(t) \\ c(t) = H_c x(t) + Lg(t) + L_d d(t) \end{cases} \tag{1}$$

where: $t \in \mathbb{Z}$, $x \in \mathbb{R}^n$ is the state vector (which includes the controller states under dynamic regulation), $g \in \mathbb{R}^m$ the manipulable reference vector which, if no constraints (and no CG) were present, would coincide with the desired reference $r \in \mathbb{R}^m$ and $y \in \mathbb{R}^m$ the output vector which is required to track r. The vector $d \in \mathbb{R}^{n_d}$ is a disturbance signal assumed to belong to the convex and compact set $\mathcal{D} \subset \mathbb{R}^{n_d}$:

$$d(t) \in \mathcal{D}, \quad \forall t \in \mathbb{Z}_+. \tag{2}$$

It is also assumed that $0_{n_d} \in \mathcal{D}$. Finally, $c \in \mathbb{R}^{n_c}$ represents the constrained vector which has to fulfill the set-membership constraints

$$c(t) \in \mathcal{C}, \quad \forall t \in \mathbb{Z}_+, \tag{3}$$

regardless of any possible admissible disturbance sequence realization $d(\cdot) \in \mathcal{D}$, with $\mathcal{C}$ being a prescribed convex and compact set with non-empty interior. It is further assumed that:

A1. The overall system (1) is asymptotically stable.
A2. System (1) is off-set free, i.e. $H_y(I_n - \Phi)^{-1}G = I_m$.

Roughly speaking, the CG design problem we want to solve is that of determining, at each time step t, a suitably modified reference signal $g(t)$ which is the best approximation of $r(t)$ compatible with the constraints, such that its application never produces constraint violations along the system evolutions induced by the CG commands, i.e. $c(t) \in \mathcal{C}, \forall t \in \mathbb{Z}_+, \forall d(\cdot) \in \mathcal{D}$. Classical solutions of the above stated CG design problem (see Bemporad, Casavola, & Mosca, 1997; Casavola et al., 2000) have been achieved by finding, at each time t, a CG action $g(t)$ as a function of the current reference $r(t)$ and measured state $x(t)$

$$g(t) := g(r(t), x(t)) \tag{4}$$

such that $g(t)$ is the best approximation of $r(t)$ under the condition $c(t) \in \mathcal{C}, \forall d(\cdot) \in \mathcal{D}$. Here we will focus on a slightly different approach to the CG design problem in which no measure of the state vector is required to determine $g(t)$. Such an approach, hereafter referred to as *Feed-Forward* CG (FF–CG), will be described in next sections and will assume the form

$$g(t) = \underline{g}(r(t), g(t-1), \rho(t)) \tag{5}$$

where $g(t-1)$ is the last applied command, $\rho(t)$ is a scalar whose meaning will be clarified later and $g(t)$ is the best approximation of $r(t)$ to be computed so as to ensure constraints satisfaction along the system virtual evolutions.

3. The standard and the Feed-Forward Command Governor approaches

3.1. The standard Command Governor approach

In order to make precise statements and comparisons with existing techniques and introduce the relevant notation, we describe first the classic CG approach presented in Bemporad et al. (1997), Casavola et al. (2000) and depicted in Fig. 2, which makes explicit use of the measure of the state.

Consider the constrained closed-loop system (1)–(3) satisfying assumptions A1–A2. Let the state be measurable at each time instant and consider the CG design problem as formulated in (4). As a first step let us define the following set recursions

$$\mathcal{C}_k := \mathcal{C} \sim \Delta_k, \quad k > 0, \qquad \mathcal{C}_\infty := \mathcal{C} \sim \Delta_\infty \tag{6}$$

where

$$\Delta_0 = L_d\mathcal{D}, \quad \Delta_k = \Delta_{k-1} + H_c\Phi^{k-1}G_d\mathcal{D}. \tag{7}$$

In (6) and (7), for given sets $\mathcal{A}, \mathcal{E} \subset \mathbb{R}^n$, $\mathcal{A} \sim \mathcal{E}$ denotes the Pontryagin set difference defined as $\mathcal{A} \sim \mathcal{E} := \{a : a + e \in \mathcal{A}, \forall e \in \mathcal{E}\}$ and $\mathcal{A} + \mathcal{E}$ the Pontryagin–Minkowski set sum, i.e. $\mathcal{A} + \mathcal{B} = \{a + b | a \in \mathcal{A}, b \in \mathcal{B}\}$. Moreover, Δ_∞ is the Hausdorff limit of the set sequence Δ_k. Because of asymptotic stability of Φ, it can be proved (see Gilbert & Kolmanovsky, 1999) that Δ_∞ is convex and compact and that, provided all $\mathcal{C}_k$ are non-empty, they are also convex and compact, satisfy the nesting condition $\mathcal{C}_k \subset \mathcal{C}_{k-1}$ and make $\mathcal{C}_\infty$ a nonempty convex and compact set.

Let us now introduce the set-valued future prediction (*virtual evolutions*) of the c-variable along the *virtual time* k under a constant *virtual command* $g(k) \equiv g$ and for all possible disturbance

sequence realizations $\{d(l) \in \mathcal{D}\}_{l=0}^{k}$ from the initial state x (at virtual time $k = 0$)

$$c(k, x, g, d(\cdot)) = \bigcup_{d(\cdot)\in\mathcal{D}} \left\{ H_c\left(\Phi^k x + \sum_{i=0}^{k-1} \Phi^{k-i-1}(Gg + G_d d(i))\right) + Lg + L_d d(k) \right\}. \tag{8}$$

By linearity, the latter can be rewritten as the sum of two terms: $c(k, x, g, d(\cdot)) = \bar{c}(k, x, g) + \tilde{c}(k, d(\cdot))$, where

$$\bar{c}(k, x, g) = H_c\left(\Phi^k x + \sum_{i=0}^{k-1} \Phi^{k-i-1} G g\right) + Lg \tag{9}$$

represents the disturbance-free evolution of the c-variable along the *virtual time* k under a constant *virtual command* $g(k) \equiv g$ and initial state x and

$$\tilde{c}(k, d(\cdot)) := \bigcup_{d(\cdot)\in\mathcal{D}} \left\{ \sum_{i=0}^{k-1} H_c \Phi^{k-i-1} G_d d(i) + L_d d(k) \right\} \subseteq \Delta_k, \tag{10}$$

the set-valued virtual evolutions due to all possible disturbance sequence realizations, all contained in Δ_k. As a consequence we have

$$\begin{array}{c} \bar{c}(k, x, g) \in \mathcal{C}_k, \quad \forall k \in \mathbb{Z}_+ \\ \Downarrow \\ c(k, x, g, d(\cdot)) = \bar{c}(k, x, g) + \tilde{c}(k, d(\cdot)) \subset \mathcal{C}, \quad \forall k \in \mathbb{Z}_+. \end{array} \tag{11}$$

Thus, constraints fulfillment can be ensured by only considering the disturbance-free evolutions of the system (1) and adopting a "worst-case" approach. To this end, let us introduce, for a given sufficiently small scalar $\delta > 0$, the sets:

$$\mathcal{C}^\delta := \mathcal{C}_\infty \sim \mathcal{B}_\delta, \qquad \mathcal{W}^\delta := \{g \in \mathbb{R}^m : c_g \in \mathcal{C}^\delta\} \tag{12}$$

where $\mathcal{B}_\delta$ is the ball of radius δ centered at the origin and $\mathcal{W}^\delta$, which we assume non-empty, the closed and convex set of all constant commands g whose corresponding disturbance-free equilibrium points $c_g := H_c x_g + Lg$ satisfy the constraints with margin δ, being $x_g := (I_n - \Phi)^{-1} G g$.

The main idea behind the CG is to choose at each time step a constant virtual command $g(\cdot) \equiv g$, with $g \in \mathcal{W}^\delta$, such that the corresponding virtual evolutions fulfill the constraints over a semi-infinite horizon $k \in [0, \infty)$ and its "distance" from the constant reference of value $r(t)$, valuated by $\|g - r(t)\|_\Psi^2$, $\Psi = \Psi^T > 0$, is minimal. Such a command is applied, a new state is measured and the procedure is repeated at the next time instant. Formally, if we introduce the set of all admissible virtual sequences from initial state x

$$\mathcal{V}(x) = \{g \in \mathcal{W}^\delta : \bar{c}(k, x, g) \in \mathcal{C}_k, \forall k \in \mathcal{Z}_+\} \tag{13}$$

the CG will choose a command $g(t)$ at each time instant t as the solution of the following constrained optimization problem

$$g(t) = \arg \min_{g\in\mathcal{V}(x(t))} \|g - r(t)\|_\Psi^2. \tag{14}$$

The following Proposition 1 summarizes the properties of the standard CG algorithm:

Proposition 1 (*Bemporad et al., 1997; Casavola et al., 2000*)**.** *Let assumptions* A1–A2 *be fulfilled. Consider system* (1)–(3) *and let* $\mathcal{V}(x(0))$ *be non-empty. Then:*

(1) *The minimizer in* (14) *uniquely exists at each* $t \in \mathbb{Z}_+$ *and can be obtained by solving a convex constrained optimization problem, viz.* $\mathcal{V}(x(0))$ *non-empty implies* $\mathcal{V}(x(t))$ *non-empty along the trajectories generated by the CG command* (14)*.*

(2) *The set* $\mathcal{V}(x), \forall x \in \mathbb{R}^n$*, is finitely determined, viz. there exists an a priori known integer* k_0 *(see* Gilbert & Tin-Tan, 1991*) such that if* $\bar{c}(k, x, w) \in \mathcal{C}_k$, $k \in \{0, 1, \ldots, k_0\}$*, then* $\bar{c}(k, x, w) \in \mathcal{C}_k \forall k \in \mathbb{Z}_+$.

(3) *The constraints are fulfilled for all* $t \in \mathbb{Z}_+$.

(4) *The overall system is asymptotically stable; in particular, whenever* $r(t) \equiv r, g(t)$ *converges in finite time either to* r *or to its best steady-state admissible approximation,* $g(t) \to \hat{r} := \arg\min_{g\in\mathcal{W}^\delta} \|g - r\|_\Psi^2$. □

3.2. The Feed-Forward Command Governor (FF–CG) approach

Here the goal is to present a different approach to the CG problem that enables us to deal with the case that no state measurements are available to the CG. The idea underlying such an approach consists in ensuring that any admissible variation of the manipulated reference $g(\cdot)$ always produces a guaranteed bounded perturbation on the actual constrained vector c around a suitable feasible steady-state value. Such a property can be ensured by opportunely limiting the reference variations by means of the following technical expedients:

(1) the computation of a new FF–CG action $g(t)$ is performed every τ steps, τ being a suitable integer to be determined, rather than at each time instant t as in the standard CG approach. Moreover, each new FF–CG command is applied for exactly τ steps;

(2) the displacement between the new FF–CG command $g(t)$ and the previous one $g(t - \tau)$ is explicitly bounded during the FF–CG computation, i.e. $g(t) - g(t - \tau) \in \Delta\mathcal{G}(g(t - \tau), \rho(t - \tau))$,

where the integer $\tau > 0$ and the closed and convex set $\Delta\mathcal{G}(g, \rho) \subset \mathbb{R}^m$ are determined from the outset whereas $\rho(t)$ is a time-varying scalar parameter selected on-line. The definitions and usage of τ, $\Delta\mathcal{G}(g, \rho)$ and $\rho(\cdot)$ will be introduced and discussed later.

The first difference with the standard CG design method is in the way the effect of the disturbances is taken into account. In fact, while in the standard CG approach the *virtual evolutions* $\bar{c}(k, x, g)$ are computed on the basis of the measured state $x(t)$ and the effect of disturbances on the predictions can be exactly characterized along the prediction horizon, here such information is not available and disturbances have to be taken into account as if they were acting on each element of the sequence $\bar{c}(k, x, g)$ from an arbitrarily long time in the past. In order to deal with this problem, denote $x(t) = \hat{x}(t) + \tilde{x}(t)$ where $\hat{x}$ is the disturbance-free component of the state and $\tilde{x}$ depending only on the disturbances. Then, $\bar{c}(k, x, g)$ can be rewritten as the sum of three terms

$$c(k, x, g, d(\cdot)) = \bar{c}(k, \hat{x}, g) + \tilde{c}(k, d(\cdot)) + H_c \Phi^k \tilde{x}. \tag{15}$$

Note that if no measure of x is available, $\tilde{x}$ is not known punctually but is a set that depends on all possible past values of the disturbance $d(\cdot) \in \mathcal{D}$. It can be characterized as

$$H_c \Phi^k \tilde{x} = \bigcup_{d(\cdot)\in\mathcal{D}} \left\{ H_c \Phi^k \sum_{i=-\infty}^{0} \Phi^{-i} G_d d(i-1) \right\} \subseteq \sum_{i=k}^{\infty} H_c \Phi^i G_d \mathcal{D}. \tag{16}$$

By recalling Eq. (10), we know that

$$\tilde{c}(k, d(\cdot)) \subseteq L_d \mathcal{D} + \sum_{i=0}^{k-1} H_c \Phi^i G_d \mathcal{D}$$

which implies that the following inclusion holds true

$$\tilde{c}(k, d(\cdot)) + H_c \Phi^k \tilde{x} \subseteq \Delta_\infty. \tag{17}$$

It finally results that $\bar{c}(k, \hat{x}, g) \in \mathcal{C}_\infty, \forall k \in \mathbb{Z}_+ \implies c(k, x, g, d(\cdot)) \subset \mathcal{C}, \forall k \in \mathbb{Z}_+$. As a consequence, in the present context, the constraints fulfillment is obtained by ensuring that

$$\bar{c}(k, \hat{x}, g) \in \mathcal{C}_\infty, \quad \forall k \in \mathbb{Z}_+. \tag{18}$$

Such a condition can be further simplified by manipulating the *virtual evolutions* $\bar{c}(k, x, g)$ as follows

$$\bar{c}(k, \hat{x}, g) = c_g + H_c \Phi^k (\hat{x} - x_g) \tag{19}$$

where x_g and c_g represent the steady-state values of the state and constrained vector and $H_c \Phi^k (\hat{x} - x_g)$ the constrained vector transient evolution. Like in the standard CG solution, we will restrict our attention to virtual commands g contained in $\mathcal{W}^\delta$. Then, the steady-state component of the virtual evolutions will always belong to $\mathcal{C}^\delta$, viz. $c_g \in \mathcal{C}^\delta$. As depicted in Fig. 3, a sufficient condition to ensure that the constraints will be satisfied, although in a quite arbitrary and conservative way, is that of ensuring that the vanishing transient component of $\bar{c}$ is confined into a ball of radius ρ_g

$$\|H_c \Phi^k (\hat{x} - x_g)\| \leq \rho_g, \quad \forall k \geq 0 \tag{20}$$

where ρ_g represents the minimum distance between c_g and the border of $\mathcal{C}_\infty$. Such a quantity can be computed by solving the following simple optimization problem

$$\begin{aligned} &\rho_g := \arg\max_{\rho} \rho \\ &\text{subject to } \mathcal{B}_\rho(c_g) \subseteq \mathcal{C}_\infty. \end{aligned} \tag{21}$$

where $\mathcal{B}_\rho(c_g)$ represents the ball of radius ρ centered in c_g. Please note that, by construction, $\rho_g \geq \delta, \forall g \in \mathcal{W}^\delta$. Details on the solution of (21) are reported in Section 4.2. Then, the key idea behind the construction of an effective FF–CG algorithm is as follows: let us decide to modify the FF–CG command signal only every τ steps and assume that, at time $t - \tau$, a command $g(t - \tau) \in \mathcal{W}^\delta$ has been computed such that the transient component of $\bar{c}(k, \hat{x}(t - \tau), g(t - \tau))$ is confined in a ball of known radius $\rho(t - \tau)$ (which is also contained in the ball of radius $\rho_{g(t-\tau)}$). This makes the condition (20) hold true, i.e.

$$\|H_c \Phi^k (\hat{x}(t - \tau) - x_{g(t-\tau)})\| \leq \rho(t - \tau) \leq \rho_{g(t-\tau)}, \quad \forall k \geq 0. \tag{22}$$

If we build an algorithm which, on the basis of the above information, were able to select at time t a new command $g(t) \in \mathcal{W}^\delta$ and a scalar $\rho(t) \geq 0$ such that the transient components of $\bar{c}(k, \hat{x}(t), g(t))$ were confined within a ball of radius $\rho(t) \leq \rho_{g(t)}$, then the constraints would be again satisfied, i.e.

$$\|H_c \Phi^k (\hat{x}(t) - x_{g(t)})\| \leq \rho(t) \leq \rho_{g(t)}, \quad \forall k \geq 0 \tag{23}$$

and, by mathematical induction, a FF–CG command $g(t)$ would be proved to exist at each time instant $t \in \mathbb{Z}_+$ provided that it would exist at time $t = 0$. Before going on, it is worth remarking that $\rho(t)$ and ρ_g represent two distinct quantities: namely $\rho(t)$ represents the actual "radius" of a ball that contains the future transient evolution of $\bar{c}$, while ρ_g is the maximum value (depending on g) such a radius may assume without violating constraints. It is important to separate the two concepts because in many cases $\rho(t) \ll \rho_g$. Let us concentrate now on how the term $\|H_c \Phi^k (\hat{x}(t) - x_{g(t)})\|$ can be bounded. The key observation here is that, if we wait for a sufficiently long time after the application of a new FF–CG command, the transient contribution decreases and its 2-norm reduces of a certain fraction of its initial value. More formally, let us introduce the following notion of *Generalized Settling Time*:

Definition (*Generalized Settling Time*)**.** The integer $\tau > 0$ is said to be a *Generalized Settling Time with parameter* γ, with $0 < \gamma < 1$, for the pair (H_c, Φ), if

$$\begin{aligned} &\|H_c \Phi^k x\| \leq M(x), \quad \forall k = 0, 1, \ldots, \tau - 1 \\ &\qquad \Downarrow \\ &\|H_c \Phi^{\tau+k} x\| \leq \gamma M(x), \quad \forall k \geq 0 \end{aligned} \tag{24}$$

holds true for each $x \in \mathbb{R}^n$, with the real $M(x) > 0$ any upper-bound to $\|H_c \Phi^k x\|, \forall k \geq 0$. □

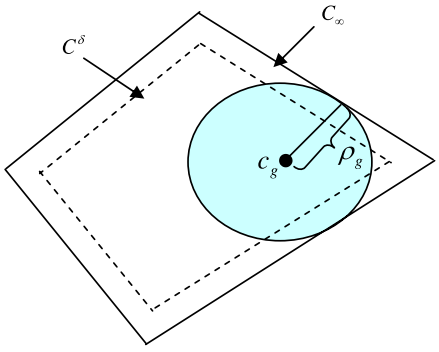

Fig. 3. Geometrical representation of condition (20) for $c \in \mathbb{R}^2$.

As a consequence, if the time interval between two command variations τ were chosen to be a generalized settling time with parameter $\gamma \in (0, 1)$ and $g(t-\tau) = g(t-\tau+1) = \cdots = g(t-1)$, the disturbance free c-transient would be bounded as follows

$$\|H_c \Phi^k (\hat{x}(t) - x_{g(t-\tau)})\| \leq \gamma \rho(t - \tau), \quad \forall k \geq 0, \tag{25}$$

because $\Phi^\tau (\hat{x}(t - \tau) - x_{g(t-\tau)}) = (\hat{x}(t) - x_{g(t-\tau)})$. The latter observation can be used to characterize the set of all possible feasible commands which may be applied at time t complying with the constraint (23). To this end, let us parameterize the generic, non necessarily feasible, command $g \in \mathcal{W}^\delta$ to be applied to the system at time t as the sum of the previously applied command $g(t - \tau)$ and of a command increment Δg to be determined by the FF–CG, i.e. $g = g(t - \tau) + \Delta g$. By noticing that $x_{\Delta g} = x_g - x_{g(t-\tau)}$ and by exploiting the triangular inequality, we can easily bound the transient component of $\bar{c}(k, \hat{x}(t), g)$ as follows

$$\begin{aligned} &\|H_c \Phi^k (\hat{x}(t) - x_g)\| && (26)\\ &\quad = \|H_c \Phi^k (\hat{x}(t) - x_{g(t-\tau)}) - H_c \Phi^k x_{\Delta g}\| && (27)\\ &\quad \leq \|H_c \Phi^k (\hat{x}(t) - x_{g(t-\tau)})\| + \|H_c \Phi^k x_{\Delta g}\| && (28)\\ &\quad \leq \gamma \rho(t - \tau) + \|H_c \Phi^k x_{\Delta g}\| \leq && (29)\\ &\quad \leq \rho(t), \quad \forall k \geq 0. && (30) \end{aligned}$$

Then, a possible way to define the scalar $\rho(t)$ at time t complying with (30) is:

$$\rho(t) := \gamma \rho(t - \tau) + \max_{k \geq 0} \|H_c \Phi^k x_{\Delta g}\| \tag{31}$$

where such a bound, as will be clearer in the next section, is finitely determinable. Moreover, note that it does not depend on the state, but only on $\rho(t-\tau) \leq \rho_{g(t-\tau)}$ and on the free command increment Δg, which has to be determined so that (23) holds true. By direct examination, the latter requirement simply consists of selecting Δg such that $\rho(t) \leq \rho_g$. In fact, the last condition can be explicitly rewritten as follows

$$\|H_c \Phi^k x_{\Delta g}\| \leq \rho_{g(t-\tau)+\Delta g} - \gamma \rho(t - \tau), \quad \forall k \geq 0. \tag{32}$$

Finally, by recalling the static map $x_{\Delta g} = (I - \Phi)^{-1} G \Delta g$, the set of all feasible FF–CG commands at time t can be characterized as

$$\begin{cases} g \in \mathcal{W}^\delta \\ (g - g(t - \tau)) \in \Delta \mathcal{G}(g(t - \tau), \quad \rho(t - \tau)) \end{cases} \tag{33}$$

where $\Delta\mathcal{G}(g,\rho)$ is defined as the set of all τ-step incremental commands from $g(t-\tau)$ ensuring inequality (32) to hold true:

$$\Delta\mathcal{G}(g,\rho) = \{\Delta g : \|H_c\Phi^k(I-\Phi)^{-1}G\Delta g\| \le \rho_{g+\Delta g} - \gamma\rho, \forall k \ge 0\}. \quad (34)$$

The properties of the above set will be detailed in Section 4 where it will be shown that it is finitely determinable ((32) have to be explicitly checked only for $0 \le k \le k_0$, where k_0 is a certain integer $k_0 \le \tau$) convex and a closed set. Moreover it will be proved that it is nonempty because it always contains a ball of radius $\eta^* > 0$, $\Delta\mathcal{G}(g,\rho) \supseteq \mathcal{B}_{\eta^*}$ for any possible command $g \in \mathcal{W}^\delta$ and for any scalar $\rho : 0 \le \rho \le \rho_g$. Finally, by using the same selection index of the standard CG approach, we can formulate the Feed-Forward CG algorithm as follows

The FF–CG algorithm

REPEAT AT EACH TIME $t = k\tau,\ k = 1, 2, \ldots$.

1.1 SOLVE

$$g(t) = \arg\min_g \|g - r(t)\|_\Psi^2 \quad (35)$$

SUBJECT TO (33)

1.2 APPLY $g(t)$ FOR THE NEXT τ STEPS

1.3 UPDATE $\rho(t) = \gamma\rho(t-\tau) + \max_{k\ge 0}\|H_c\Phi^k(I-\Phi)^{-1}G\Delta g(t)\|$.

It is possible to prove that, under the assumption that a feasible couple $(g(0), \rho(0))$ complying with (23) is known at time $t = 0$ (which also implies constraints specifications and disturbance set $\mathcal{D}$ make $\mathcal{W}^\delta$ and $\mathcal{G}(g(0),\rho(0))$ non-empty), the above FF–CG scheme enjoys the following properties:

Theorem 1. *Let assumptions* A1–A2 *be fulfilled. Consider system (1) along with the FF–CG selection rule and let an admissible command signal $g(0) \in \mathcal{W}^\delta$ be applied at $t = 0$ and a scalar $\rho(0)$ exist such that $\|H_c\Phi^k(x(0) - x_{g(0)})\| \le \rho(0) \le \rho_{g(0)}, \forall k \ge 0$. Then:*

(1) *At each decision time $t = k\tau, k \in \mathbb{Z}_+$, the minimizer in (35) uniquely exists and can be obtained by solving a convex constrained optimization problem;*

(2) *The system acted by the FF–CG never violates the constraints, i.e. $c(t) \in \mathcal{C}$ for all $t \in \mathbb{Z}_+$ regardless of any possible admissible disturbance realization $d(\cdot) \in \mathcal{D}$. Namely at each time step such a vector will satisfy:*

$$c(t) \in (c_{g(t)} + \mathcal{B}_{\rho(t)}) + \Delta_\infty, \quad \forall t \in \mathbb{Z}_+ \quad (36)$$

where the term $(c_{g(t)} + \mathcal{B}_{\rho(t)})$ is a ball of center $c_{g(t)}$ and radius $\rho(t) < \rho_{g(t)}$, $g(t) = g\left(\left\lfloor\frac{t}{\tau}\right\rfloor\tau\right)$ and $\rho(t) = \rho\left(\left\lfloor\frac{t}{\tau}\right\rfloor\tau\right)$. $\lfloor\cdot\rfloor$ denotes the standard floor operator;

(3) *The disturbance-free state $\hat{x}(t)$ lies in a convex and closed set centered in the steady-state value $x_{g(t-1)}$, i.e.*

$$\hat{x}(t) \in (x_{g(t-1)} + \mathcal{X}_{\rho(t)},\ \forall t \in \mathbb{Z}_+) \quad (37)$$

where

$$\mathcal{X}_\rho = \{x \in \mathbb{R}^n | \|H_c\Phi^k x\| \le \rho, k = 0, \ldots, \infty\}. \quad (38)$$

Moreover, if (H_c, Φ) is observable, $\mathcal{X}_\rho$ is compact.

(4) *Whenever $r(t) \equiv r$, with r a constant set-point, the sequence of $g(t)$'s converges in finite time either to r or to its best admissible steady-state approximation $\hat{r}$, i.e. $\exists t' > 0$ such that*

$$g(t) = \hat{r} := \arg\min_{g\in\mathcal{W}^\delta} \|g - r\|_\Psi^2, \quad \forall t \ge t' \quad (39)$$

and moreover

$$\lim_{t\to\infty}\hat{x}(t) = x_{\hat{r}},\ \lim_{t\to\infty}\hat{y}(t) = y_{\hat{r}} = \hat{r}. \quad (40)$$

Proof. (1) The existence of an admissible solution at each decision time $k\tau$ can be proved by simply remarking that $g(t) = g(t-\tau)$, to be chosen along with $\rho(t) = \gamma\rho(t-\tau)$, is always an admissible, although not necessarily the optimal, solution for the prescribed problem at time t. Moreover, being the admissible region (33) the intersection of two convex regions $\mathcal{W}^\delta$ and $\Delta\mathcal{G}(g,\rho)$ it results in a convex set as well. Then, a minimizer for the constrained optimization problem (35) subject to (33) always exists and its uniqueness follows from the strict convexity of its cost function and admissible region.

(2) At each decision time $t = k\tau$, with $k \in \mathbb{Z}_+$, a command $g(k\tau)$ complying with (33) is applied to the plant. By construction, the latter implies that the set-valued virtual predictions along the virtual time i defined in (8) satisfy $c(i, x(k\tau), g(k\tau), d(\cdot)) \subseteq \mathcal{C}, \forall d(\cdot) \in \mathcal{D}, \forall i \in \mathbb{Z}_+$. Then, the statement is proved by simply noticing that the inclusion $c(t) \in c(i, x(k\tau), g(k\tau), d(\cdot)) \subseteq \mathcal{C}, \forall d(\cdot) \in \mathcal{D}$ holds true for all time instants $t = k\tau + i, i \in \{0, 1, \ldots, \tau-1\}$ and by repeating the same argument for all $k \in \mathbb{Z}_+$. Moreover, it can be observed that, under the FF–CG action $g(k\tau)$, $\hat{c}(t) = \bar{c}(i, \hat{x}(k\tau), g(k\tau))$, for $t = k\tau + i, \forall i \in \{0, 1, \ldots, \tau-1\}$. Then, the fulfillment of conditions (33) implies, via Eq. (23), that $\|\bar{c}(i, \hat{x}(k\tau), g(k\tau)) - c_{g(k\tau)}\| \le \rho(k\tau), \forall i \in \mathbb{Z}_+$, which, by combining the latter with (15) and (17), implies (36).

(3) Conditions (37) and (38) simply follow by noting that, thanks to (23), the following bound holds true

$$\begin{aligned}&\|H_c\Phi^{k+i}(\hat{x}(t) - x_{g(t)})\| \\ &= \|H_c\Phi^k(\hat{x}(t+i) - x_{g(t)})\| \\ &\le \rho(t), \quad i = 0, \ldots, \tau-1, k = 0, \ldots, \infty. \end{aligned} \quad (41)$$

The convexity of the set $\mathcal{X}(\rho)$ can be proved by means of classical triangular inequalities arguments. In fact, because

$$\begin{aligned}&\|H_c\Phi^k(\lambda x' + (1-\lambda)x'')\| \\ &= \|\lambda H_c\Phi^k x' + (1-\lambda)H_c\Phi^k x''\| \\ &\le \lambda\|H_c\Phi^k x'\| + (1-\lambda)\|H_c\Phi^k x''\|\end{aligned}$$

one can state that if $x', x'' \in \mathcal{X}(\rho)$ for a fixed value of ρ, then $(\lambda x' + (1-\lambda)x'') \in \mathcal{X}(\rho), \forall\lambda \in [0\ 1]$. Finally, consider the first n conditions $\|H_c\Phi^k x\| \le \rho,\ k = 0, 1, \ldots, n-1$ of (38). They define a compact set inside the image of the linear map $c_x = \Theta x$ with Θ the Observability matrix

$$\Theta := [(H_c)^T (H_c\Phi)^T (H_c\Phi^2)^T \cdots (H_c\Phi^{n-1})^T]^T$$

and

$$c_x := [(H_c x)^T, (H_c\Phi x)^T, \ldots, (H_c\Phi^{n-1}x)^T]^T. \quad (42)$$

Whenever the pair (H_c, Φ) is completely observable, the rank of Θ is full and coincides with its column-rank n. In this case, the linear map between x and c_x is injective and admits the left-inverse operator $x = (\Theta^T\Theta)^{-1}\Theta^T c_x$. As a consequence, any compact set inside the image of Θ has a compact pre-image on x.

(4) Consider $r(t) = r, \forall t \ge t^*$ and assume, without loss of generality, $\Psi = I$. Let $g(t)$ be the FF–CG action at time t, solution of the optimization problem (35) with optimal cost $V^2(t) = \|g(t) - r\|^2$. Clearly, $V(t)$ corresponds to the minimal Euclidean distance between $g(t)$ and r compatible with the constraints. As already stated, at time $t+\tau, g(t)$ is still an admissible, though not necessarily the optimal, solution at time $t+1$. Hence $V(t+\tau) = \|g(t+\tau) - r\| \le \|g(t) - r\| = V(t)$. Thus $V(t)$, for constant set-points $r(t) \equiv r$, is a monotonically not increasing sequence. In order to prove the finite-time convergence of this succession to $\hat{r}$, it is enough to evaluate the two possible exclusive situations:

1. if the optimal $g(t+\tau)$ is such that $\Delta g(t+\tau)$ belongs to the interior of $\Delta\mathcal{G}(g(t),\rho(t))$, i.e. $\Delta g(t+\tau) \in \text{In}[\Delta\mathcal{G}(g(t),\rho(t))]$, then no points $g \in \mathcal{W}^\delta$ belonging to any

sufficiently small neighborhood $g \in \Xi$ of $g(t+\tau)$ would give rise to a lower cost $\|g - r\| < \|g(t+\tau) - r\|$, for all $g \in \Xi$. This implies that, because of convexity of the objective function (35), $g(t+\tau) = \hat{r}$.

2. if $g(t+\tau)$ is such that $\Delta g(t+\tau)$ belongs to the border of $\Delta\mathcal{G}(g(t), \rho(t))$, then, because of Proposition 2's results, the distance between the old command $g(t)$ and the new command $g(t+\tau)$ is at least η^*. This implies that $V(t+\tau)$ will at least decrease by either η^*, in the case $g(t+\tau) \in \mathrm{In}[\mathcal{W}^\delta]$, or a fixed and computable amount $\Delta V_{\min}(r) > 0$ in the case $g(t+\tau)$ belongs also to the border of $\mathcal{W}^\delta$.

The latter allows one to conclude that for any $r \in \mathbb{R}^m$ there exists an integer

$$t' \leq \tau \left\lceil \frac{V(0) - \hat{V}_r}{\Delta V_{\min}(r)} \right\rceil \tag{43}$$

with $\lceil\cdot\rceil$ denoting the ceiling operator and $\hat{V}_r := \|\hat{r} - r\|$. Finally, conditions (40) follow directly from A1–A2. □

A complete characterization of the properties and computation the generalized settling time τ and of the convex set $\Delta\mathcal{G}(g, \rho)$ will be provided in Section 4. Moreover, in the last subsection, we will show how those machineries can be fruitfully optimized for the special (but very common) case of box constraints.

4. Technical details

4.1. Generalized settling time τ

In order to build an efficient FF–CG algorithm it is important to determine a good estimate of the minimum generalized settling time τ for a given factor γ. As a first step, let us recall the following obvious result proved in Gilbert and Tin-Tan (1991):

Lemma 1. *Let Φ asymptotically stable. Then, for any $x \in \mathbb{R}^n$ an integer $k_0 \in \mathbb{Z}_+$ exists such that $\|H_c\Phi^k x\| \leq M$, $k = 0, \ldots, k_0$, implies $\|H_c\Phi^k x\| \leq M$, $k = k_0 + 1, \ldots, \infty$ for a whatever sufficiently large positive real $M \in \mathbb{R}_+$.* □

As already noticed, such a result makes many of the quantities (e.g. $\mathcal{G}(g, \rho)$) of the above FF–CG scheme finitely determinable once the integer k_0 is determined, e.g. via the algorithm provided in Gilbert and Tin-Tan (1991). Interestingly enough such an integer k_0 can be seen as a particular case of a generalized settling time when $\gamma = 1$. The following result shows, for the general case of $\gamma < 1$, the existence of an analytically determinable upper-bound to the minimum generalized settling time:

Lemma 2. *Let the pair (H_c^o, Φ^o) be the observable subsystem obtained via a canonical Kalman observability decomposition of (H_c, Φ) and Φ be asymptotically stable. Then, the integer*

$$\bar{\tau} = \left\lceil \log_\lambda \left(\frac{\gamma}{\bar{\sigma}(H_c^o) B \bar{\sigma}\{((\Theta^o)^T\Theta^o)^{-1}(\Theta^o)^T\}\sqrt{n}} \right) \right\rceil \tag{44}$$

provides an upper-bound to the minimum generalized settling time τ with parameter γ for (H_c, Φ), where $\Theta^o := [(H_c^o)^T, (H_c^o\Phi^o)^T, \ldots, (H_c^o(\Phi^o)^{n-1})^T]^T$ and B and λ two scalars such that $\|(\Phi^o)^k\| \leq B\lambda^k$.

Proof. First observe that the constraints are influenced only by the observable part of the state x^o, with $x^o = P^o x$ and P^o accounting for the change of state coordinates defined by the Kalman observability decomposition. In fact, one has that $H_c\Phi^k x = H_c^o(\Phi^o)^k x^o$, $\forall k \in \mathbb{Z}_+$. Moreover, the asymptotic stability of Φ (and hence of Φ^o) implies the exponential convergence of its modes. Then, there exist two computable positive scalars $B > 0$, $\lambda \in (0, 1)$ such that $\|(\Phi^o)^k\| \leq B\lambda^k$. Our goal is to find an integer τ such that if

$$\|H_c^o(\Phi^o)^k x^o\| \leq M(x^o), \quad k = 0, \ldots, \infty \tag{45}$$

then $\|H_c^o(\Phi^o)^{k+\tau} x^o\| \leq \gamma M(x^o)$, $k = 0, \ldots, \infty$ follows. By resorting to the notation and results of Theorem 1, we can prove that $\|x^o\| = \|((\Theta^o)^T\Theta^o)^{-1}(\Theta^o)^T c_x\| \leq \bar{\sigma}\{((\Theta^o)^T\Theta^o)^{-1}(\Theta^o)^T\}\|c_x\|$, where c_x is the vector defined in (42) and $\bar{\sigma}\{((\Theta^o)^T\Theta^o)^{-1}(\Theta^o)^T\}$ denotes the largest singular value of $((\Theta^o)^T\Theta^o)^{-1}(\Theta^o)^T$. Under the hypothesis (45) holding true, we can easily show that $\|c_x\|^2 = \|H_c^o x^o\|^2 + \|H_c^o\Phi^o x^o\|^2 + \cdots + \|H_c^o(\Phi^o)^{n-1}x^o\|^2 \leq nM^2(x^o)$ and, because $\|c_x\| \leq \sqrt{n}M(x^o)$, we can bound $\|x^o\|$ as $\|x^o\| \leq \bar{\sigma}\{((\Theta^o)^T\Theta^o)^{-1}(\Theta^o)^T\}\sqrt{n}M(x^o)$. Then, by exploiting the exponential stability of Φ^o, the term $\|H_c^o(\Phi^o)^k x^o\|$ can be bounded as

$$\|H_c^o(\Phi^o)^k x^o\| \leq \bar{\sigma}(H_c^o) B\lambda^k \bar{\sigma}\{((\Theta^o)^T\Theta^o)^{-1}(\Theta^o)^T\}\sqrt{n}M(x^o) \tag{46}$$

and an upper-bound to the minimum generalized settling time can be determined by looking for the minimum integer τ such that

$$\bar{\sigma}(H_c^o) B\lambda^\tau \bar{\sigma}\{((\Theta^o)^T\Theta^o)^{-1}(\Theta^o)^T\}M(x^o)\sqrt{n} \leq \gamma M(x^o).$$

Finally, (44) is obtained by resorting to the logarithms' properties. □

The settling time obtained by means of the latter result might be conservative. Hereafter we propose a method to find tighter τ complying with (24). The first observation is that, for our goals, we do not need to verify (24) for all $x \in \mathbb{R}^n$, but we can restrict our attention to a finitely determinable convex and closed set

$$\begin{aligned}\Omega &:= \{x \in \mathbb{R}^n : \|H_c\Phi^k x\| \leq \rho_{\max}, k = 0, \ldots, \infty\} \\ &= \{x \in \mathbb{R}^n : \|H_c\Phi^k x\| \leq \rho_{\max}, k = 0, \ldots, k_0\}\end{aligned}$$

where k_0 complying with Lemma 1 is computed as in Gilbert and Tin-Tan (1991) and $\rho_{\max} = \max_{g \in \mathcal{W}^\delta} \rho_g$ represents the maximum distance ρ_g between the steady-state value c_g and the border of $\mathcal{C}_\infty$ obtainable for any admissible input vector $g \in \mathcal{W}^\delta$. Then, our task is to find an integer τ ensuring that the satisfaction of

$$\|H_c\Phi^k x\| \leq M(x), \quad \forall x \in \Omega, \forall k = 0, \ldots, \tau - 1 \tag{47}$$

for some $M(x)$ implies that $\|H_c\Phi^{\tau+i}x\| \leq \gamma M(x), \forall x \in \Omega, \forall i = 0, \ldots, k_0$ are satisfied as well. Consider now the function $M'(x) := \max_{k=0,\ldots,\tau-1} \|H_c\Phi^k x\|$. By construction, such a function is a lower-bound for any possible $M(x)$ complying with (47). Because $M'(x) \leq M(x)$, we can reformulate the computation of the generalized settling time as the problem of finding an integer τ such that

$$\|H_c\Phi^{\tau+i}x\| \leq \gamma \max_{k=0,\ldots,\tau-1} \|H_c\Phi^k x\|, \quad \forall x \in \Omega, \forall i = 0, \ldots, k_0.$$

A necessary and sufficient condition for the latter is that there exists an integer $k \in \{0, \ldots, \tau - 1\}$ such that $\gamma\|H_c\Phi^k x\| - \|H_c\Phi^{\tau+i}x\| \geq 0, \forall x \in \Omega, \forall i = 0, \ldots, k_0$. Then, our goal can be rewritten as the problem of finding a τ such that there exists a $k \in \{0, \ldots, \tau - 1\}$ ensuring

$$\min_{x \in \Omega}(\gamma\|H_c\Phi^k x\| - \|H_c\Phi^{\tau+i}x\|) \geq 0, \quad \forall i = 0, \ldots, k_0. \tag{48}$$

The latter reformulation can be used to build up an algorithm for the determination of the minimum τ by performing a binary search in the range $[1, \bar{\tau}]$. Note that even such an algorithm is typically computationally onerous, it is not unrealistic to solve it off-line from the outset. Furthermore, it is worth to remark that even if problem (48) is non-convex, it relies in the special class of the Difference of Convex functions programming (a.k.a. DC programming) problems and can be thus solved in an efficient way (see Horst & Thoai, 1999).

4.2. Computation and properties of $\Delta\mathcal{G}(g, \rho)$

The goal of this subsection is to completely characterize the set $\Delta\mathcal{G}(g, \rho)$ when $\mathcal{C}_\infty$ consists of polyhedral constraints, which are

representable as a collection of linear inequalities

$$\mathcal{C}_\infty := \{c \in \mathbb{R}^{n_c} : Tc \le q\} \tag{49}$$

where $T = [T_1, \dots, T_{n_v}]^T \in \mathbb{R}^{n_v \times n_c}$, $q = [q_1, \dots, q_{n_v}]^T \in \mathbb{R}^{n_v}$ and the operator $\le$ acts component-wise. By resorting to its definition in (34) and by exploiting Lemma 1, the set $\Delta\mathcal{G}(g, \rho)$ consists of all vectors Δg such that inequalities $\|H_c\Phi^k(I - \Phi)^{-1}G\Delta g\| \le \rho_{g+\Delta g} - \gamma\rho$ hold true for $k = 0, \dots, k_0$. The first step in order to understand the meaning of the above inequalities is to make clear what the term $\rho_{g+\Delta g}$ is. By definition (21), the latter represents the minimum Euclidean distance between $c_{g+\Delta g}$ and the border of the admissible set $\mathcal{C}_\infty$. The admissible set being a polyhedron, such a distance can be computed as the minimum of the Euclidean distances between the point $c_{g+\Delta g}$ and each of the hyperplanes $T_i^T c = q_i, i = 1, \dots, n_v$ defining the half-spaces $T_i^T c \le q_i, i = 1, \dots, n_v$. By using standard geometrical results, such a distance can be computed as the solution of the following linear programming problem

$$\rho_{g+\Delta g} = \max_{\bar\rho} \bar\rho \tag{50}$$

$$\text{subject to: } 0 \le \bar\rho \le \frac{q_i - T_i^T c_{g+\Delta g}}{\|T_i\|}, \quad i = 1, \dots, n_v \tag{51}$$

where the scalar $\bar\rho$ is a slack variable and the right-most term in (51) describes the standard unsigned Euclidean distance between $c_{g+\Delta g}$ and the i-th hyperplane describing the boundaries of the polyhedral set $\mathcal{C}_\infty$. Because $c_{g+\Delta g} \in \mathcal{C}_\infty$, such a term is always non-negative. The latter expression allows one to describe the set $\Delta\mathcal{G}(g, \rho)$ as the set of vectors Δg such that a positive scalar $\bar\rho \ge \gamma\rho$ exists ensuring

$$\|H_c\Phi^k(I - \Phi)^{-1}G\Delta g\| \le \bar\rho - \gamma\rho, \quad k = 0, \dots, k_0 \tag{52}$$

$$\bar\rho \le \frac{q_i - T_i^T[H_c(I - \Phi)^{-1}G(g + \Delta g) + L(g + \Delta g)]}{\|T_i\|}, \quad i = 1, \dots, n_v. \tag{53}$$

By means of the latter reformulation, the following properties may be proved.

Proposition 2. *Let the set $\Delta\mathcal{G}(g, \rho)$ be characterized by (52)–(53) for any $g \in \mathcal{W}^\delta$ and $\rho \in \mathbb{R}$ such that, $0 \le \rho \le \rho_g$. Then it is:*

1. *convex, closed and finitely determinable;*
2. *nonempty: in particular there exists a scalar $\eta^* > 0$ such that the set $\Delta\mathcal{G}(g, \rho)$ contains a ball of radius η^*, i.e. $\exists\eta^* > 0 : \Delta\mathcal{G}(g, \rho) \supseteq \mathcal{B}_{\eta^*}, \forall g \in \mathcal{W}^\delta, \forall\rho \le \rho_g, \rho \ge 0$;*
3. *compact whenever $\operatorname{rank}\{G^o\} = m$, where the tuple (H_c^o, Φ^o, G^o) is the observable subsystem obtained via a canonical Kalman observability decomposition of (H_c, Φ, G).*

Proof. (1) Closure and finite determinateness follow by the fact that the set $\Delta\mathcal{G}(g, \rho)$ may be computed by means of the finite set of non-strict inequalities (52)–(53). Regarding convexity, let us assume two vectors $[\Delta g_1^T, \bar\rho_1]^T$ and $[\Delta g_2^T, \bar\rho_2]^T$ are given complying with (52)–(53). We need to prove that, for any $\lambda \in [0, 1]$, the vector $[\lambda\Delta g_1^T + (1 - \lambda)\Delta g_2^T, \lambda\bar\rho_1 + (1 - \lambda)\bar\rho_2]^T$ still complies with (52)–(53). This may be trivially verified for the linear inequalities (53). Let us focus on the left hand side of (52). By the triangular inequality, we obtain

$$\begin{aligned}&\|H_c\Phi^k(I - \Phi)^{-1}G(\lambda\Delta g_1 + (1 - \lambda)\Delta g_2)\|\\ &= \|\lambda H_c\Phi^k(I - \Phi)^{-1}G\Delta g_1 + (1 - \lambda)H_c\Phi^k(I - \Phi)^{-1}G\Delta g_2\|\\ &\le \lambda\|H_c\Phi^k(I - \Phi)^{-1}G\Delta g_1\| + (1 - \lambda)\|H_c\Phi^k(I - \Phi)^{-1}G\Delta g_2\|,\\ &\quad k = 0, \dots, k_0.\end{aligned}$$

Then, because $[\Delta g_1^T, \bar\rho_1]^T$ and $[\Delta g_2^T, \bar\rho_2]^T$ satisfy (52), for $\lambda \ge 0$, $(1 - \lambda) \ge 0$ we obtain

$$\begin{aligned}&\|H_c\Phi^k(I - \Phi)^{-1}G(\lambda\Delta g_1 + (1 - \lambda)\Delta g_2)\|\\ &\le \lambda(\bar\rho_1 - \gamma\rho) + (1 - \lambda)(\bar\rho_2 - \gamma\rho)\\ &= (\lambda\bar\rho_1 + (1 - \lambda)\bar\rho_2) - \gamma\rho,\end{aligned}$$

for all $k = 0, \dots, k_0$ which finally ensures

$$\begin{aligned}&\|H_c\Phi^k(I - \Phi)^{-1}G(\lambda\Delta g_1 + (1 - \lambda)\Delta g_2)\|\\ &\le (\lambda\bar\rho_1 + (1 - \lambda)\bar\rho_2) - \gamma\rho, \quad k = 0, \dots, k_0.\end{aligned}$$

(2) If we denote the Euclidean distance between the point c_g and the i-th support hyperplane of $\mathcal{C}_\infty$ as $\rho_{g,i} = \frac{q_i(T_i^T(H_c(I-\Phi)^{-1}G+L)g)}{\|T_i\|}$, $i = 1, \dots, n_v$ we can rewrite the inequality (53) as

$$\bar\rho \le \rho_{g,i} - \frac{T_i^T(H_c(I - \Phi)^{-1}G + L)\Delta g}{\|T_i\|}, \quad i = 1, \dots, n_v. \tag{54}$$

By combining (54) with (52), also rewriting $\Delta\mathcal{G}(g, \rho)$ as the set of vectors Δg such that

$$\begin{aligned}&\|H_c\Phi^k(I - \Phi)^{-1}G\Delta g\|\\ &\le \rho_{g,i} - \frac{T_i^T(H_c(I - \Phi)^{-1}G + L)\Delta g}{\|T_i\|} - \gamma\rho,\\ &\quad k \ge 0,\ i = 1, \dots, n_v.\end{aligned} \tag{55}$$

Then, because $\rho_g \ge \rho$, a sufficient condition ensuring (55) holding true is given by

$$\begin{aligned}&\|H_c\Phi^k(I - \Phi)^{-1}G\Delta g\|\\ &\le -\frac{T_i^T(H_c(I - \Phi)^{-1}G + L)\Delta g}{\|T_i\|} + \rho_{g,i} - \gamma\rho_g,\\ &\quad k \ge 0,\ i = 1, \dots, n_v.\end{aligned} \tag{56}$$

Let us focus now on the term $\rho_{g,i} - \gamma\rho_g$. By construction $\rho_g \le \rho_{g,i}$. Then, $\rho_{g,i} - \gamma\rho_g \ge (1 - \gamma)\rho_g \ge 0$. Moreover, $g \in \mathcal{W}^\delta$ implies that $\rho_g \ge \delta$. Then, $\rho_{g,i} - \gamma\rho_g \ge (1 - \gamma)\delta$ and the condition (56) can be rewritten as

$$\begin{aligned}&\|H_c\Phi^k(I - \Phi)^{-1}G\Delta g\| + \frac{|T_i^T(H_c(I - \Phi)^{-1}G + L)\Delta g|}{\|T_i\|}\\ &\le (1 - \gamma)\delta, \quad k \ge 0, i = 1, \dots, n_v.\end{aligned} \tag{57}$$

Now, if we decompose the FF–CG action increment $\Delta g \in \mathbb{R}^m$ as the product of an arbitrary unitary vector $\hat v \in \mathbb{R}^m(\|\hat v\| = 1)$ and of scalar $\eta > 0$, such that $\Delta g = \hat v\eta$, condition (57) becomes

$$\begin{aligned}&\eta\left(\|H_c\Phi^k(I - \Phi)^{-1}G\hat v\| + \frac{|T_i^T(H_c(I - \Phi)^{-1}G + L)\hat v|}{\|T_i\|}\right)\\ &\le (1 - \gamma)\delta, \quad k \ge 0, i = 1, \dots, n_v.\end{aligned} \tag{58}$$

To prove the claim it is enough to note that, for each possible $k \ge 0, i = 1, \dots, n_v$ and for any possible direction $\hat v$, the term $\left(\|H_c\Phi^k(I - \Phi)^{-1}G\hat v\| + \frac{|T_i^T(H_c(I-\Phi)^{-1}G+L)\hat v|}{\|T_i\|}\right)$ is bounded. Then, there exists a sufficiently small value η^* such that inequality (58) holds true for any possible direction vector $\hat v$ and for any $\eta \le \eta^*$.

(3) In order to prove compactness consider that, C_∞ being a compact set, the scalar $\rho_{\max} \ge \rho_g, \forall g \in \mathcal{W}^\delta$ is always bounded. Then, a necessary condition for (52) being satisfied is $\|H_c\Phi^k(I - \Phi)^{-1}G\Delta g\| \le \rho_{\max}, k \ge 0$, which is equivalent to $\|H_c^o(\Phi^o)^k(I - \Phi^o)^{-1}G^o\Delta g\| \le \rho_{\max}, k \ge 0$. By resorting to the same arguments used in Theorem 1, the latter implies that for any admissible $\Delta g \in \Delta\mathcal{G}(g(t - \tau), \rho(t - \tau))$ the vector $v = (I - \Phi^o)^{-1}G^o\Delta g$ belongs to a set of observable states which can be shown to be compact with the same arguments used in the proof of item 2 of Theorem 1. This

finally implies that all Δg inverse-images of such v have to belong to a compact set as well if the matrix $(I - \Phi^o)^{-1}G^o$ is full column rank, which requires $\text{rank}\{G^o\} = m$, being $(I - \Phi^o)^{-1}$ invertible by assumption. □

4.3. The special case with box constraints

In many practical applications constraints may be represented by box constraints in the form

$$\mathcal{C}_\infty := \{c \in \mathbb{R}^{n^c} : \underline{q}_i \le c_i \le \bar{q}_i, i = 1, \ldots, n_c.\} \tag{59}$$

with $\underline{q}_i \in \mathbb{R}$ and $\bar{q}_i \in \mathbb{R}$ representing respectively the maximum and the minimum value that the i-th component of the constrained vector c may assume. By exploiting this particular constraint structure it is possible to simplify the design of the presented FF–CG and reduce its conservativeness. The main insight is that we can easily decompose the constraint set $\mathcal{C}_\infty$ into the Cartesian product of n_c sets of scalar intervals:

$$\mathcal{C}_\infty = \mathcal{C}^{(1)} \times \mathcal{C}^{(2)} \times \cdots \times \mathcal{C}^{(n_c)} \tag{60}$$

where $\mathcal{C}^{(i)} := \{c_i \in \mathbb{R} | \underline{q}_i \le c_i \le \bar{q}_i\}$. While this structure does not give any benefit in the computation of the set $\mathcal{W}^\delta$, it may give advantages in the computation of the command increment Δg. The idea is that, if we choose Δg_i such that $\hat{c}_i(k, \hat{x}(t), g_i(t)) \in \mathcal{C}^{(i)}$, $k \ge 0$ for all $i = 1, \ldots, n_c$, then $\bar{c}(k, \hat{x}(t), g(t)) \in \mathcal{C}_\infty$, $k \ge 0$. This can be easily exploited by means of the following algorithm:

The FF–CG algorithm (box constraints)

REPEAT AT EACH TIME $t = \kappa\tau, \kappa = 1, 2, \ldots$

1.1 SOLVE

$$g(t) = \arg\min_g \|g - r(t)\|^2_\Psi \tag{61}$$

SUBJECT TO

$$\begin{cases} g \in \mathcal{W}^\delta \\ (g_i - g_i(t-\tau)) \in \Delta\mathcal{G}^{(i)}(g(t-\tau), \rho^{(i)}(t-\tau)), \\ i = 1, \ldots, n_c. \end{cases} \tag{62}$$

1.2 APPLY $g(t)$.

1.3 UPDATE $\rho^{(i)}(t) = \gamma\rho^{(i)}(t-\tau) + \max_{k\ge 0}\{|H_{c,i}\Phi^k(I-\Phi)^{-1}G\Delta g|\}$, $i = 1, \ldots, n_c$

where $H_{c,i}$ represents the i-th line of H_c and sets $\Delta\mathcal{G}^{(i)}(g, \rho^{(i)})$ are defined as

$$\Delta\mathcal{G}^{(i)}(g, \rho^{(i)}) := \{\Delta g \in \mathbb{R}^n : |H_{c,i}\Phi^k(I-\Phi)^{-1}G\Delta g| \le \rho^{(i)}_{g+\Delta g} - \gamma\rho^{(i)}, k \ge 0\}.$$

Note that, the Euclidean distance being defined as $\rho^{(i)}_{g+\Delta g} = \min\{c_{g+\Delta g} - \underline{q}_i, \bar{q}_i - c_{g+\Delta g}\}$ those sets are very easily computable. Finally, note that also the computation of the generalized settling time with parameter γ can be simplified in this case. In fact, it can be computed as the largest of all settling times obtained for each single constrained variable, i.e. $\tau = \max_{i=1,\ldots,n_c}\{\tau^{(i)}\}$ where $\tau^{(i)}$ is the settling time with parameter γ for the couple $(\Phi, H_{c,i})$.

5. Illustrative example: position servomechanism

Consider the position servomechanism schematically described in Fig. 4. It consists of a DC-motor, a gear-box, an elastic shaft and an uncertain load. A detailed description is given in Bemporad, Casavola, and Mosca (1998). Let θ_M and θ_L denote respectively the motor and the load angles and let the state be $x = [\theta_L, \dot{\theta}_L, \theta_M, \dot{\theta}_M]^T$. Because of the finite shear strength of the steel shaft, a maximum admissible shaft torque of $|T| \le 78.5398$ N m is imposed as a constraint on the torsional torque. Moreover, it is assumed that

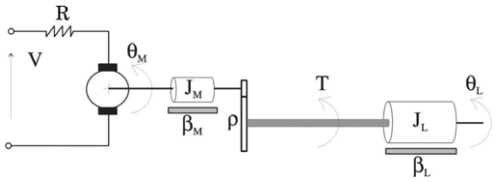

Fig. 4. Servomechanism model.

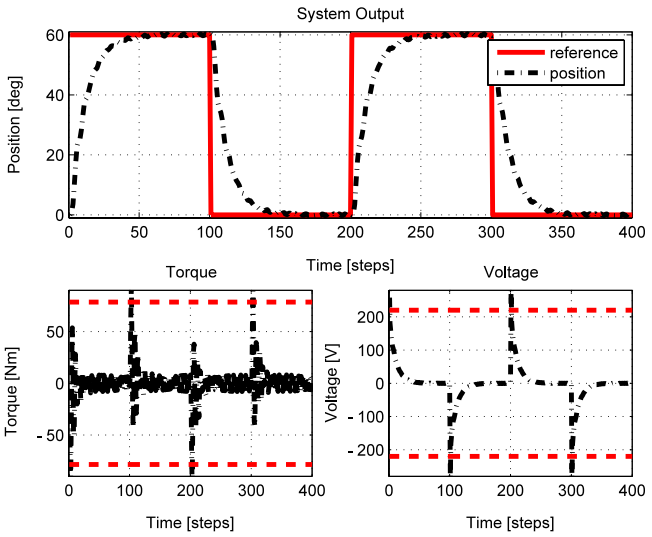

Fig. 5. No-disturbance case, system without CG unit: (upper) Position; (lower) Constrained variables: Torsional Torque (left), voltage (right).

the input DC voltage V has to be constrained within the range $|V| \le 220\,V$. Please note that the two constraints are independent and form a boxed region. It is assumed that the plant is pre-compensated so as to guarantee assumptions A1–A2. Here the signal $g(t)$ represents the reference position. Finally it is assumed that at time $t = 0$ the system is at the equilibrium, i.e. $x(0) = [0, 0, 0, 0]^T$ for $g = 0$. The presented plant, when not governed by a CG unit, exhibits a very fast response but inadmissible voltage inputs and torsional torques for the references of interest as shown in Fig. 5 (a square-wave set-point with amplitude $r = 60$ deg (solid line) and no disturbances, i.e. $d(t) = 0$). The simulations for the CG governed system are reported in Fig. 6, where the output $y(t) = \theta_L(t)$ (upper) and the computed CG commands $g(t)$ are depicted for the same set-point of Fig. 5. In this figure, the classical CG solution (CG in the figures) and both FF–CG and FF–CG(Box) methods are applied, the latter being synthesized on the basis of a generalized settling time $\tau = 7$ with contraction factor $\gamma = 0.6$. It is fair commenting that both the CG and FF–CG(Box) algorithms are built so as to exploit the fact that the system is subject to box constraints whereas the FF–CG method does not enjoy this capability and it expected to behave more conservatively. Interestingly enough, the performance of the three algorithms are very similar.

In Fig. 7, the constrained variables are depicted: in this case the input voltage and the torsional torque always satisfy the respective constraints. It is worth noticing that the trajectories of the system acted by standard CG and FF–CG(Box) are very close to each other and both are much closer to the constraint boundaries than the ones resulting under the FF–CG scheme.

In order to show the effectiveness and performance robustness of FF–CG when disturbances are present, we consider the severe case whereby $d(t)$ is a square-wave with amplitude $d_{\max} = 0.007$ N (Figs. 8 and 9). From the above figures it can be observed that both the FF–CG algorithms keep working adequately also if the state is brought far from the equilibrium by the disturbances. In particular, the constraints are always satisfied and the tracking

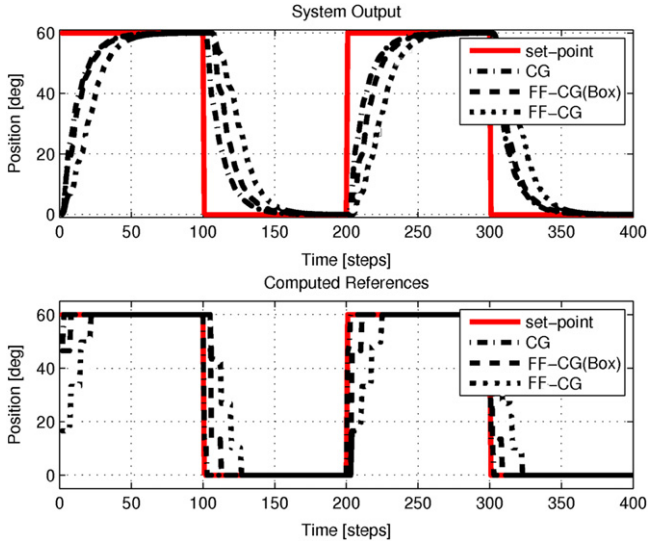

Fig. 6. No-disturbance case, system with CG and FF–CG units: (upper) System Output; (lower) Computed CG actions.

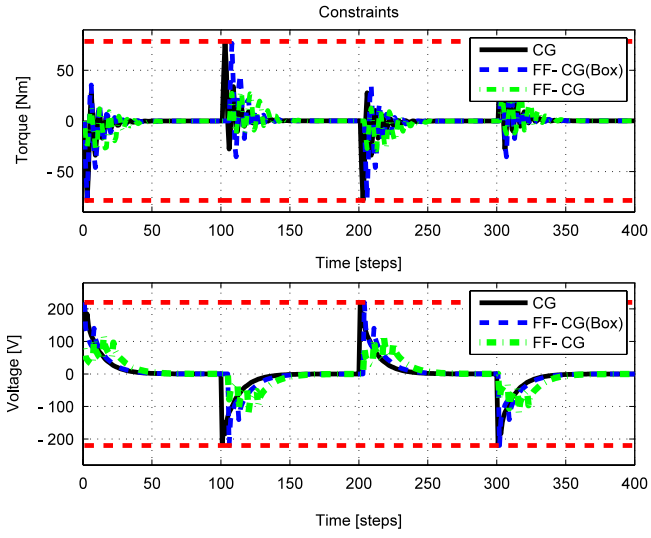

Fig. 7. No-disturbance case, constrained variables: (upper) Torque; (lower) Voltage.

performance is achieved. It is fair also pointing out that, in the simulations, the applied disturbance is unrealistically high, producing perturbation on the nominal torque as high as 40 N m, when the maximum admissible torque is 78.5398 N m. With smaller and smoother disturbance signals, the system controlled by both the FF–CG strategies presents a behavior similar to the one shown in Figs. 6 and 7.

A further analysis on the performance robustness under disturbances for all FF–CG and CG schemes has been performed and the results have been depicted in Fig. 10-(upper), where performance-degradation vs. disturbance-size plots are reported. In this respect, performance comparisons are expressed by the cost $J = \frac{1}{T}\sum_{k=0}^{T} |r(k) - \theta_L(k)|$ computed by executing simulations with square-wave disturbances characterized by different amplitudes $d_{\max}$, with T the number of steps of the simulation (in this case $T = 400$). As a result, the performance of the standard CG unit is only weakly affected by the disturbance size whereas the performance of the other two schemes monotonically degrades with the increment of the amplitude of the disturbance. Note also that for $d_{\max} > 0.01$ the standard CG design problem is unfeasible whereas the other algorithms are unfeasible for lower values:

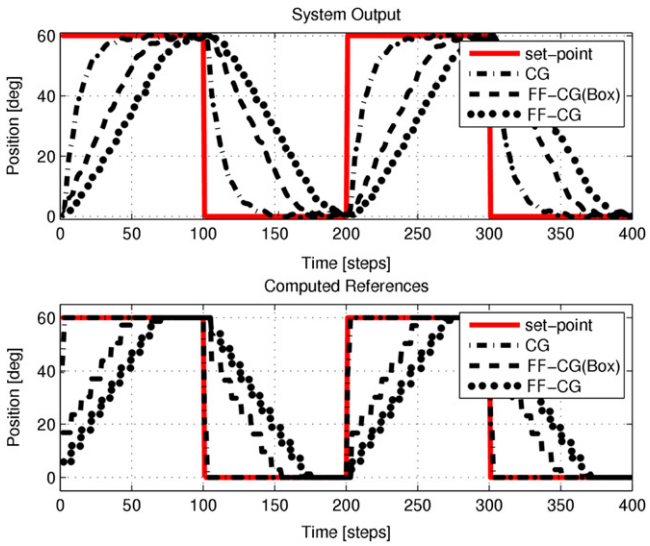

Fig. 8. Square-disturbance case, system with CG and FF–CG units: (upper) System Output; (lower) Computed CG actions.

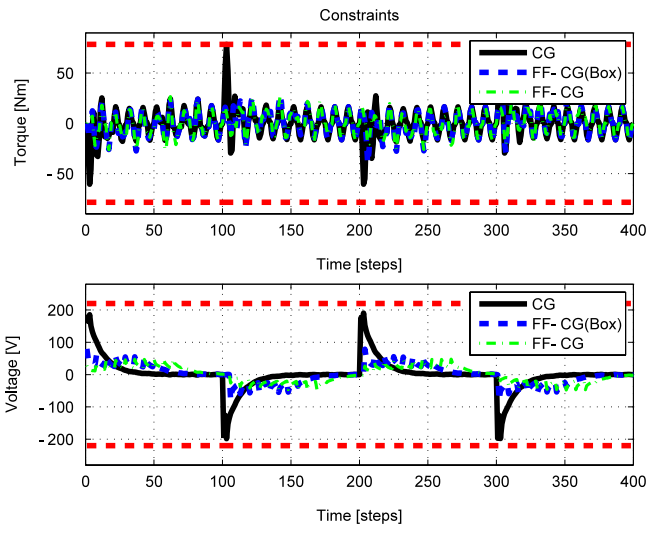

Fig. 9. Square-disturbance case, systems with CG and FF–CG units: (upper) Torque; (lower) Voltage.

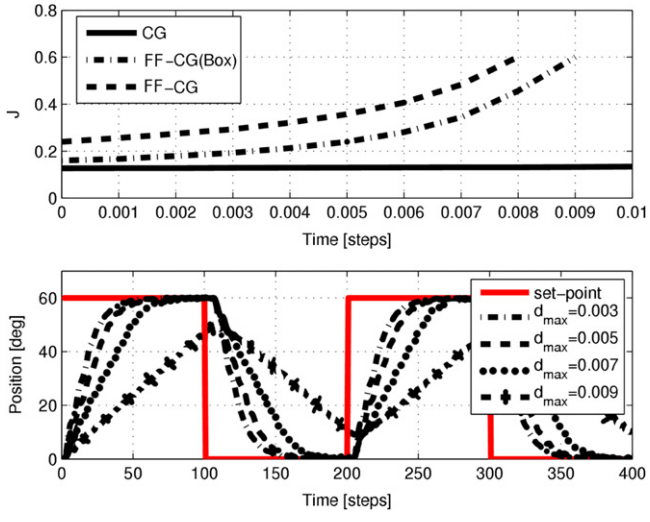

Fig. 10. (Upper) Performance degradation vs. disturbance sizes. (lower) FF–CG (Box) performance degradation vs. disturbance sizes.

FF–CG(Box) is unfeasible for $d_{\max} > 0.009$ and FF–CG for $d_{\max} > 0.008$ respectively. A different way to see performance degradation due to disturbances is depicted in Fig. 10-(lower), where the output of the system acted by the FF–CG(Box) is reported for different values for $d_{\max}$. Finally, in the following table the on-line computational burdens per step of all schemes are reported. Please note that, as expected, the FF–CG(Box) is more efficient than FF–CG because it has to solve an easier convex optimization problem than the one solved by the general FF–CG.

	CG	FF–CG (Box)	FF–CG
CPU time (ms)	40.0	37.2	69.5

6. Conclusions

In this paper, a novel CG scheme is proposed which, thanks to the asymptotic stability of the pre-compensated system, its action computation is not based on the explicit measure of the state. The main idea is to limit reference variations to always maintain the constrained variable trajectory "not too far" from the space of admissible steady-state equilibria. The properties of the proposed algorithm have been carefully analyzed and the differences with standard CG approaches pointed out. Comparisons with the classical CG solutions have been presented and commented in the final illustrative example. It is expected that this class of solutions might be of interest in all applications where either the measure or the estimation of the whole state is problematic, e.g. in decentralized or distributed networked applications where the cost to make all the entire aggregate state known to all agents at each time instant could be unrealistic or require unrealistic communication infrastructures. Studies on extending the proposed centralized FF–CG solution to decentralized and distributed networked frameworks are in progress.

References

Angeli, A., Casavola, A., & Mosca, E. (1999). Command Governors for constrained nonlinear systems: direct nonlinear vs. linearization-based strategies. *International Journal of Robust and Nonlinear Control*, *9*, 677–699.

Angeli, A., Casavola, A., & Mosca, E. (2001). On feasible set-membership state estimators in constrained Command Governor control. *Automatica*, *37*, 151–156.

Angeli, A., & Mosca, E. (1999). Command Governor for nonlinear systems under constraints. *IEEE Transactions on Automatic Control*, *44*, 816–820.

Bemporad, A. (1998). Reference governor for constrained nonlinear systems. *IEEE Transactions on Automatic Control*, *43*, 415–419.

Bemporad, A., Casavola, A., & Mosca, E. (1997). Nonlinear control of constrained linear systems via predictive reference management. *IEEE Transactions on Automatic Control*, *42*, 340–349.

Bemporad, A., Casavola, A., & Mosca, E. (1998). A predictive reference governor for constrained control systems. *Computers in Industry*, *36*, 55–64.

Bemporad, A., & Mosca, E. (1998). Fulfilling hard constraints in uncertain linear systems by reference managing. *Automatica*, *34*, 451–461.

Casavola, A., Mosca, E., & Angeli, D. (2000). Robust Command Governors for constrained linear systems. *IEEE Transactions on Automatic Control*, *45*, 2071–2077.

Casavola, A., Papini, M., & Franzè, G. (2006). Supervision of networked dynamical systems under coordination constraints. *IEEE Transactions on Automatic Control*, *51*(3), 421–437.

Cirre, C. M., Berenguel, M., Valenzuela, L., & Klempous, R. (2009). Reference governor optimization and control of a distributed solar collector field. *European Journal of Operational Research*, *193*, 709–717.

Falcone, P., Borrelli, F., Peckar, J., & Steward, G. (2009). A reference governor approach for constrained piecewise affine systems. In *Proceedings of European control conference 2009. Budapest, Hungary*.

Garone, E., Tedesco, F., & Casavola, A. (2009). Distributed coordination-by-constraint strategies for networked control systems. In *1st workshop on distributed estimation and control in networked systems. NecSys09. Venice, Italy. 24–26 September*.

Gilbert, E. G., & Kolmanovsky, I. (1999). Fast reference governors for systems with state and control constraints and disturbance inputs. *International Journal of Robust and Nonlinear Control*, *9*, 1117–1141.

Gilbert, E. G., & Kolmanovsky, I. (2001). A generalized reference governor for nonlinear systems. In *IEEE CDC*.

Gilbert, E. G., Kolmanovsky, I., & Tin Tan, K. (1995). Discrete-time reference governors and the nonlinear control of systems with state and control constraints. *International Journal of Robust and Nonlinear Control*, *5*, 487–504.

Gilbert, E. G., & Tin Tan, K. (1991). Linear system with state and control constraints: the theory and application of maximal output admissible sets. *IEEE Transaction on Automatic Control*, *36*, 1008–1020.

Hatanaka, T., & Takaba, K. (2005). Output feedback reference governor for nonlinear systems. In *Proceedings of IEEE conference on decision and control 2005. San Diego, CA, USA. December*.

Horst, R., & Thoai, N. V. (1999). DC programming overview. *Journal Optimization Theory & Applications*, *103*, 1–43.

Mattei, M., Famularo, D., & Labate, C. V. (2009). Constrained plasma shape control in iter. In *Proceedings of European control conference 2009. Budapest, Hungary. August*.

Sugie, T., & Yamamoto, H. (2001). Reference management for closed loop systems with state and control constraints. In *Proceedings of IEEE American control conference 2001. Arlington, Virginia, USA. June*.

Emanuele Garone obtained his Laurea degree from the University of Calabria, Italy in 2005 and received his Ph.D. degree in Systems Engineering from the University of Calabria, Italy, in 2009. From 2008 to 2010 he has been with the Department of Electronics, Informatics and Systems of the University of Calabria, Italy, as an Assistant Researcher. From November 2010 he is an Assistant Professor at the SAAS Department of the Université Libre de Bruxelles. His current research interests include: constrained predictive control, Linear Parameter Varying systems, control under constraints, control reconfiguration for fault tolerant systems, networked control systems, path planning for heterogeneous vehicles, control of switching systems.

Francesco Tedesco was born in Cosenza in 1984. He received the Laurea Specialistica degree in Automation Engineering in 2008 and since 2009 he is a Ph.D. student at the Department of Electronics, Informatics and Systems of the University of Calabria, Italy. During the academic year 2009/2010 he visited the Automatic Control Laboratory, ETH Zürich, Switzerland. His main research interests are networked control systems, control with constraints and supervision approaches over data network.

Alessandro Casavola was born in Florence, Italy, in 1958. He received his Ph.D. degree in Systems Engineering from the University of Bologna, Italy, in 1990. He has been with the Department of Systems and Computer Engineering of the University of Florence from 1996 to 1998 as a Researcher. Since 1998 he is with the Department of Electronics, Informatics and Systems of the University of Calabria: as an Associate Professor first and as a Full Professor since 2005. His current research interests include constrained predictive control, control under constraints, control reconfiguration for fault tolerant systems and supervision approaches over data networks. He is an Editor of the International Journal of Adaptive Control and Signal Processing.