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journal homepage: [www.elsevier.com/locate/sysconle](http://www.elsevier.com/locate/sysconle)Switching control laws in the presence of measurement noise<sup>☆</sup>Emanuele Garone<sup>a</sup>, Roberto Naldi<sup>b,\*</sup>, Emilio Frazzoli<sup>c</sup><sup>a</sup> Dipartimento di Elettronica, Informatica e Sistemistica (DEIS), Università della Calabria, I-87036 Rende (CS), Italy<sup>b</sup> Center for Research on Complex Automated Systems (CASYS), University of Bologna, Viale Risorgimento 2, I-40136 Bologna, Italy<sup>c</sup> Laboratory for Information and Decision Systems (LIDS), Massachusetts Institute of Technology, Cambridge, MA 02139, USA

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## ABSTRACT

In this paper we study a class of systems governed by piecewise continuous control laws, subject to measurement noise and such that more than one state configuration could be a desired point for the system to be stabilized. Such a problem is common in many practical situations, including, for example, vehicle attitude control, robotic manipulator positioning, deployment of multistable structures, etc. Our aim is to characterize, both for continuous and discrete-time models, control properties and switching policies ensuring that any possible state trajectory of the closed-loop system is robustly asymptotically convergent to one of the desired “target points”. First we will show that in the discrete-time case, conditions guaranteeing the asymptotic stability to a set of points can be derived by exploiting the intrinsic “holding” nature of the discrete-time framework. Such results can be extended to continuous time by means of standard sample-and-hold techniques. However, this kind of solution is revealed to be very conservative and then not realistically employable in many contexts. For such a reason further less restrictive policies will be introduced and investigated; these are mostly based on the concept of “strategy holding” in time (by providing a slower switching policy sampling time) and space (by using hysteresis regions). Those strategies will be characterized both in terms of global stabilizability and in terms of switching flexibility, i.e., the capability to react in a convenient way to sudden state changes. Driven by the above analysis an “opportunistic switching” strategy combining all the advantages of the strategies considered will be proposed and analyzed. Synthesis results are presented, providing a constructive procedure to design the proposed control laws. The paper is concluded with an application example based on the attitude control of a Ducted-Fan Aerial Vehicle, showing the effectiveness of the proposed approach in a practical application.

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## 1. Introduction

Piecewise continuous control laws (see [1–3]) are often employed in applications where the point in the state space to which the system has to be asymptotically stabilized is not defined univocally a priori, but may depend on the current state of the system. In some applications of particular interest, such as attitude stabilization (see e.g. [4]), the use of discontinuous control laws is a consequence of topological issues that make it impossible to obtain global results using continuous feedback (see [5]). In this

scenario the presence of noise, even arbitrarily small, may result in undesired switching between different control laws, which could eventually prevent the attainment of the desired control goal. Several approaches have been proposed in the literature to solve this class of problems. In particular, for continuous-time systems, “sampling-and-hold” control policies [6,7] and hysteresis [8,9] have been proposed.

In this paper, motivated by the framework introduced in [10, 11], by the previous results in [12–18] and by the interest that these kinds of systems have in real applications, we propose a detailed analysis of the above problem for both continuous and discrete time systems. First, we will show that discrete-time control formulations, unlike the continuous time counterpart, are able to achieve implicitly some convergence properties under certain conditions on the form of the control law. This is due to the implicit “1-step input holding nature” of discrete-time models that can be recovered in the continuous-time setting by means of sample-and-hold techniques. However, in both cases, such robustness is attained at the cost of very restrictive hypotheses on the control laws, depending on the measurement noise magnitude.

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Motivated by the above reason, one of the main contributions of the paper will be to propose and analyze different approaches able to guarantee the desired system behavior with less conservative conditions.

In particular, after a brief analysis of the so-called initial state dependent control laws, a class of “policy holding” strategies is proposed. The main idea of this kind of approach is to enable switches only at certain appropriately defined decision times. One of the main advantages of the proposed approach is that it introduces a decoupling property in the synthesis of the overall control strategy. In fact, it allows us to design the individual control laws (almost) regardless of the effects of the measurement noise. Global stability to the desired set of points is ensured by an appropriate choice of decision times and control law scheduling policies. A further noticeable approach we will study is based on the hysteresis on the state space and follows classical hysteresis ideas very well known in the hybrid control literature (see e.g. [8]).

The above three classes of strategies will be characterized by underlining, both in the continuous and the discrete time framework, the invariance conditions needed to guarantee global stability to a set of points. Moreover, we will investigate their behavior in the case of sudden state variations in order to evaluate the flexibility of those strategies. Such an analysis will drive the development of mixed policies, referred to as “opportunistic switching” strategies, able to merge the advantages of time-holding and state hysteresis strategies.

The above theoretical developments are complemented with synthesis methods based on invariant sets and Lyapunov arguments.

The paper is organized as follows. In Section 2 the class of systems which are considered in the paper and the control problem are introduced. In Section 3 a simple physical example is used to more intuitively explain the problem under consideration. Such an example will be carried out throughout the rest of the paper to illustrate the proposed strategies. Sections 4 and 5 investigate switching policies and contain the main results of the paper. In Sections 6 and 7 some synthesis issues are discussed and an application example regarding the attitude control of a Ducted-Fan Aerial Vehicle is reported.

## 2. Problem statement

Consider the following dynamical system subject to measurement noise:

$$\begin{aligned} \delta x(t) &= f(x(t), u(t)), \\ y(t) &= x(t) + e(t), \end{aligned} \quad (1)$$

where  $\delta$  represents time differentiation in the continuous-time case, i.e.  $\delta x(t) = \dot{x}(t)$ , and the one-step shift operator  $\delta x(t) = x(t+1)$  in the discrete time case, and:

- $x(t) \in X \subseteq \mathbb{R}^n$  is the state,
- $y(t) \in Y \subseteq \mathbb{R}^m$  is the measured output,
- $u(t) \in U \subseteq \mathbb{R}^m$  is the manipulable input,
- $e(t) \in E \subseteq \mathbb{R}^m$  is the measurement noise.

In the following, we will assume that the set  $E$  is bounded, and hence that there exists a positive real  $e_{\max}$  such that  $\sup_{e \in E} \|e\| \leq e_{\max}$ , where  $\|\cdot\|$  denotes the Euclidean norm.

Finally, it is supposed that the control system (1) is well-posed, i.e.:

- $\forall x(t) \in X, \forall \tau \geq 0, \forall u : [t, t + \tau] \rightarrow U \Rightarrow x(t + \tau) \in X$
- $\forall x(t) \in X, \forall u(t) \in U \Rightarrow x(t + 1) \in X.$

for the continuous and discrete time case, respectively.

Let us recall the following notion of stability:

**Definition 1** (ISS stability with restrictions  $S$  on the initial state [19,20]). Let us consider the system

$$\delta x = f(x(t), w(t)) \quad (2)$$

where  $x(t) \in \mathbb{R}^n$  denotes the state and  $w(t) \in \mathbb{R}^m$  is an exogenous input. Let  $\bar{x} \in \mathbb{R}^n$  be an equilibrium point of (2) in the absence of the exogenous input, i.e. such that  $f(\bar{x}, 0) = 0$  for the continuous time case, and  $f(\bar{x}, 0) = \bar{x}$  for the discrete time case, respectively. System (2) is Input to State Stable (ISS) with restriction  $S \subseteq \mathbb{R}^n$  on the initial state if there exist a class- $K$  function  $\gamma : \mathbb{R}_{\geq} \rightarrow \mathbb{R}_{\geq}$  and class- $KL$  function  $\beta : \mathbb{R}_{\geq} \times \mathbb{R}_{\geq} \rightarrow \mathbb{R}_{\geq}$  such that for all  $\|w(t)\| < w_{\max}$  and for all initial states  $x(0) \in S$

$$\|x(t) - \bar{x}\| \leq \beta(\|x(0) - \bar{x}\|, t) + \gamma(\|w(t)\|), \quad \forall t \geq 0. \quad \square (3)$$

**Remark 1.** For the sake of future derivations it is important to recall that ISS systems (see [21,10]) admit a  $K$ -asymptotic gain, i.e.

$$\limsup_{t \rightarrow \infty} \|x(t) - \bar{x}\| \leq \gamma \left( \limsup_{t \rightarrow \infty} \|w(t)\| \right) \leq \gamma(w_{\max}). \quad \square$$

The system is controlled by means of a switching control law in the form

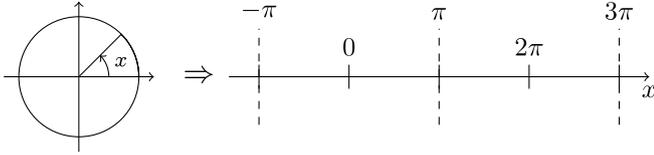
$$u(t) = \begin{cases} g_1(y(t)), & y(t) \in X_1, \\ g_2(y(t)), & y(t) \in X_2, \end{cases} \quad (4)$$

where  $\{X_1, X_2\}$  is a partition of  $X$ , i.e.,  $X_1 \cup X_2 = X, X_1 \cap X_2 = \emptyset$ , and  $g_i$  are continuous functions. Thereafter each  $g_i(y(t))$ ,  $i = 1, 2$ , is assumed to be such that the closed loop system  $\delta x(t) = f(x(t), g_i(y(t)))$  with equilibrium point  $\bar{x}_i \in X_i$  is ISS with restriction  $S_i$  on the initial state, where  $S_i$  can be any set containing all the possible values the state  $x(t)$  can assume if  $y(t) \in X_i$ , i.e.  $S_i \supseteq X_i \oplus E$  with  $X_i \oplus E = \{x \in X | \exists e \in E : y = x + e \in X_i\}$ . The above conditions in practice assure that each feedback control law  $g_i(\cdot)$ , once the initial conditions fulfill the restrictions  $S_i$ , guarantees asymptotic convergence of the system trajectories to a neighborhood of the desired equilibrium point  $\bar{x}_i$ . Restrictions have been primarily introduced to extend the results for systems in which the design of feedback laws ensuring global stability property – i.e., in which the ISS property holds for initial conditions belonging to the entire state space – is not possible or hard to achieve.

As will be shown in the next section, in the presence of measurement noise, control laws of the form (4), despite the robustness of each feedback law  $g_i(\cdot)$ , could in general give rise to limit cycles. Moreover, well-posedness issues may arise in the continuous-time case [2]. These problems mainly depend on the strategies – which will be referred to as *switching policies* – that decide *when* to apply a certain feedback law in order to converge to a certain equilibrium point.

The goal of this paper is to design, both in the continuous and the discrete-time case, conditions and switching policies able to avoid this kind of problem and then able to guarantee global practical stabilization [22] of the system to a “set of points”. In order to proceed to a systematic analysis the following stability issues will be addressed:

**Definition 2** (Initial State Dependent Stabilization). Consider the system (1), equipped with the control law (4) such that  $\bar{x}_1 \in X_1, \bar{x}_2 \in X_2$  are the two equilibrium points for  $e \equiv 0$ , then initial state dependent stabilization is ensured if for any  $y(0) \in X_i$  the closed loop trajectory  $x(t)$  converges to the equilibrium point  $\bar{x}_i$ , i.e. there



**Fig. 1.** Control of a rigid body rotating around a fixed axis. Parameterizing the configuration manifold  $S_1$  with the angle  $\theta \in \mathbb{R}$  the desired equilibrium point is mapped into an infinite sequence of values of  $\theta$ .

exist class- $K$  functions  $\gamma_1 : \mathbb{R}_{\geq} \rightarrow \mathbb{R}_{\geq}$  and  $\gamma_2 : \mathbb{R}_{\geq} \rightarrow \mathbb{R}_{\geq}$  such that

$$\begin{aligned} \limsup_{t \rightarrow \infty} \|x(t) - \bar{x}_1\| &\leq \gamma_1(e_{\max}), \quad \forall x(0) \in X_1 \\ \limsup_{t \rightarrow \infty} \|x(t) - \bar{x}_2\| &\leq \gamma_2(e_{\max}), \quad \forall x(0) \in X_2. \quad \square \end{aligned} \quad (5)$$

**Definition 3 (Global Set-stabilization).** The system (1), equipped with the control law (4) such that  $\bar{x}_1 \in X_1, \bar{x}_2 \in X_2$  are the two equilibrium points for  $e \equiv 0$ , is said to be globally stable to the set of points  $\{\bar{x}_1, \bar{x}_2\}$  if,  $\forall x(0) \in X$ , the closed loop trajectory converges indifferently to one of the equilibrium points  $\bar{x}_1$  or  $\bar{x}_2$ , i.e. there exist class- $K$  functions  $\gamma_1 : \mathbb{R}_{\geq} \rightarrow \mathbb{R}_{\geq}$  and  $\gamma_2 : \mathbb{R}_{\geq} \rightarrow \mathbb{R}_{\geq}$  such that

$$\begin{aligned} &\left( \limsup_{t \rightarrow \infty} \|x(t) - \bar{x}_1\| \leq \gamma_1(e_{\max}) \right) \\ &\vee \left( \limsup_{t \rightarrow \infty} \|x(t) - \bar{x}_2\| \leq \gamma_2(e_{\max}) \right), \quad \forall x(0) \in X. \quad \square \end{aligned} \quad (6)$$

**Remark 2.** Note that initial state dependent stabilization is a sufficient condition for global set-stabilization. Such a stronger definition has been introduced here to develop the analysis in a rigorous way and to show further interesting properties besides global set-stabilization.  $\square$

**Remark 3.** Throughout this paper, for simplicity, only the case of strategies switching between two control laws is considered. Results can be generalized to strategies switching between  $k$  control laws by the introduction of an adjacency graph.  $\square$

### 3. An attitude control example

The goal of this section is to give insight, by means of a physical example, of the design problems that can arise by the use of a control law in the form (4). To this end let us consider a discrete-time mechanical system consisting of a rigid body rotating around a fixed axis (see Fig. 1). Let us assume that the input of the system is the angular velocity. The configuration space of the system is the unit circle  $S_1 = \{z \in \mathbb{R}^2 \mid \|z\| = 1\}$ . Moreover, let us assume that the measurement of the angular position  $y(t)$  is affected by noise of bounded amplitude  $|e| < e_{\max}$  (see Fig. 2).

The overall system could be rewritten in the following way:

$$\begin{aligned} x(t+1) &= x(t) + u(t), \\ y(t) &= x(t) + e(t), \end{aligned} \quad (7)$$

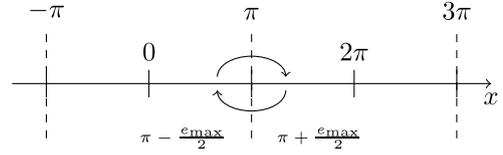
where  $x(t)$  is the angular position, used to parameterize the configuration manifold  $S_1$  and  $u(t)$  represents the angular velocity. Clearly  $x \in \mathbb{R}$  covers infinitely many times the unit circle.

In order to stabilize the system to one of the desired equilibrium points that corresponds to  $x = 2k\pi$ , with  $k \in \mathbb{Z}$ , we choose the following discontinuous output feedback law (see also [5]):

$$u_k(t) = -K(y(t) - 2k\pi) \quad \text{if } y(t) \in X_k, \quad (8)$$

where

$$X_k : x \in ((2k-1)\pi; (2k+1)\pi].$$



**Fig. 2.** Undesired limit cycle behavior arising from the use of discontinuous control in the presence of even arbitrarily small measurement noise.

Clearly, the set  $\{X_k : k \in \mathbb{N}\}$  is a partition of the real axis. It is possible to prove that, for a fixed  $k$ , the control law (8) with  $0 < K < 1$  is such that the overall closed loop system

$$x(t+1) = x(t) - K(x(t) + e(t) - 2k\pi)$$

is ISS, according to Definition 3, with equilibrium point  $2k\pi$  and without restrictions on the initial conditions.

In the general case, even if each control law (8) is ISS, the system could not converge to any of the points on  $\mathbb{R}$  mapping the desired position we want to reach. In fact, for some stabilizing control laws, it is possible to construct a measurement error sequence such that a system with the shown control law never leaves a neighborhood of the border points  $(2k+1)\pi$ .

For instance, consider  $K = e_{\max}/(\pi - 0.5 e_{\max})$  with  $|e(t)| \leq e_{\max}$  and consider the following sequence:

**Time 0**

$$\begin{aligned} x(0) &= \pi + 0.5 e_{\max} \\ e(0) &= -e_{\max} \\ y(0) &= \pi - 0.5 e_{\max} \end{aligned}$$

**Time 1**

$$\begin{aligned} x(1) &= \pi + 0.5 e_{\max} - K(\pi - 0.5 e_{\max}) \\ &= \pi + 0.5 e_{\max} - \frac{e_{\max}}{\pi - 0.5 e_{\max}}(\pi - 0.5 e_{\max}) \\ &= \pi - 0.5 e_{\max} \\ e(1) &= e_{\max} \\ y(1) &= \pi + 0.5 e_{\max} \end{aligned}$$

**Time 2**

$$\begin{aligned} x(2) &= \pi - 0.5 e_{\max} - K(\pi + 0.5 e_{\max} - 2\pi) \\ &= \pi - 0.5 e_{\max} + \frac{e_{\max}}{\pi - 0.5 e_{\max}}(\pi - 0.5 e_{\max}) \\ &= \pi + 0.5 e_{\max}. \end{aligned}$$

Because  $x(2) = x(0)$ , the sequence of measurement errors

$$e(k) = \begin{cases} -e_{\max} & k = 0, 2, 4, \dots \\ e_{\max} & k = 1, 3, 5, \dots \end{cases}$$

is such that the system continues to switch indefinitely between the two control laws without converging to any of the two equilibrium points. Such evidence on a simple but significant example shows the importance of studying conditions and strategies that avoid such a kind of limit cycle to arise in order to ensure global stability. Similar examples can be built also for continuous time systems [5,4] where, moreover, well-posedness issues arise [6].

### 4. Switching laws

In this section we will study the stability properties of systems controlled by switching control laws of the form (4). As already remarked this kind of control law may fail to achieve global stabilization to a set of points even for arbitrary small noise and in the continuous time case give rise to well-posedness issues (see, among others, [8] for details). Here we will show that, interestingly enough, such a property is achievable by means of simple invariance conditions in the discrete-time framework. To this end, let us define:

**Definition 4** (Robust Positive Invariance [23]). Given system (1), the set  $X_p \subseteq X$  is said to be “robustly positively invariant” with respect to a certain control law  $u(t) = g(y(t))$  if, for all initial conditions such that  $y(0) \in X_p$ , the output  $y(k)$  belongs to the set  $X_p$  for all  $k \in \mathbb{N}$ . □

The latter means that the set  $X_p$  is robustly positively invariant if given  $y(t) \in X_p$ , then  $y(t+1) = f(y(t) - e(t), g(y(t))) + e(t+1)$  belongs to  $X_p$ , for all possible  $e(t), e(t+1) \in E$ . The main idea is that, if a set  $X_i$  over which (4) is defined is robustly positively invariant for the feedback law  $g_i(y(t))$ , then, if  $y(t) \in X_i$  at a certain time  $t$ , the law will never switch and the overall system will converge to a neighborhood of  $\bar{x}_i$ . Such an insight allows us to characterize the initial state dependent stabilization and the global stabilization to a set of points as follows:

**Theorem 5.** Let the system (1) and a piecewise continuous control law in the form (4) be given. Moreover, let  $g_i(y(t))$  be such that the closed-loop system  $x(t+1) = f(x(t), g_i(y(t)))$  with equilibrium point  $\bar{x}_i$  is ISS with restriction  $S_i$  on the initial state, with  $X_i \subseteq X_i \oplus E \subseteq S_i$ , and  $\bar{x}_i \in X_i$  and let  $\varepsilon_i = \{x : \|x - \bar{x}_i\| \leq \gamma_i(e_{\max})\}$  for  $i = 1, 2$ . Then the following holds

- initial state dependent stabilization is ensured if  $X_i$  is a robust positively invariant set with respect to  $g_i(y(t))$ ,  $i = 1, 2$ ; the latter is also a necessary condition if  $\varepsilon_1 \cap \varepsilon_2 = \emptyset$ ;
- global set stabilization to the set of points  $\{\bar{x}_1, \bar{x}_2\}$  is ensured if  $X_1$  or  $X_2$  is a robust invariant set with respect to, respectively,  $g_1(y(t))$  or  $g_2(y(t))$ .

**Proof.** Let us prove the first item of the theorem. Given a certain  $y(0) \in X_i$ , at the first step the applied control law is  $g_i(y(t))$  and, because of the invariance condition, it will be applied for all the future steps. From the ISS property, if the control law never switches, the system asymptotically converges to a neighborhood of  $\bar{x}_i$  and then initial state dependent stabilization is achieved. Let us assume also that  $\varepsilon_1 \cap \varepsilon_2 = \emptyset$ . Let us suppose there exists a certain  $y(0) \in X_i$  such that, applying (4), the system converges asymptotically to a neighborhood of  $\bar{x}_i$  and, at a certain time  $k$ ,  $y(k) = y' \in X_j$  with  $i \neq j$ . Let there now be an initial state  $y(0) = y' \in X_j$ , which should converge to a neighborhood of  $\bar{x}_j$ . If the neighborhoods of  $\bar{x}_i$  and  $\bar{x}_j$  are disjoint this ends the proof of the first item by contradiction.

The second item of the theorem can be proved as follows. Let  $y(t) \in X_i$ . Because the applied control laws are ISS on some set  $S_i \supseteq X_i$   $i = 1, 2$ , if the law  $g_i(y(t))$  is not robustly invariant then, either  $y(t)$  converges to  $\bar{x}_i$  or there exists a finite time  $t' > 0$ , such that  $y(t') < \infty$  and  $y(t') \in X_j$ . In the latter case, since  $X_j$  is invariant,  $y(t)$  will converge to  $\varepsilon_j$ . This fact proves that global set stabilization is ensured. □

**Example.** Let us consider the example introduced in Section 2, with a certain  $|e(t)| \leq e_{\max} < \pi/2$  for all  $t \in \mathbb{N} \cup 0$  and let us define, for some  $-\pi < \varepsilon < \pi$  and for each  $k \in \mathbb{Z}$ , the sets  $X_k^*(\varepsilon) = \{x \in ((2k-1)\pi + \varepsilon, (2k+1)\pi - \varepsilon)\}$ . It is possible to rewrite the invariance condition in term of the state of the system requiring

$$x(t+1) \in X_k^*(e_{\max}), \quad \forall x(t) \in X_k^*(-e_{\max}),$$

where  $X_k^*(-e_{\max})$  denotes the set of all the states for which there exists a value of  $e(t)$ , with  $|e(t)| \leq e_{\max}$ , such that the output  $y(t)$  belongs to the set  $X_k$ , while  $X_k^*(e_{\max})$  denotes the set of all the states for which the output  $y(t)$  belongs to the set  $X_k$  for all possible values of  $e(t)$ . By means of the latter discussion it is possible to state that initial state dependent stabilization is ensured by using a law in the form (8) where the term  $K$  satisfies  $K > 2e_{\max}/\pi$ . On the contrary, if we consider a different control law:

$$u_k(t) = -K_k(y(t) - 2k\pi) \quad \forall y(t) \in X_k, \quad 0 < K_k < 1,$$

where:

$$K_k = \begin{cases} \frac{2e_{\max}}{\pi} + \varepsilon & k = 1, 3, 5, \dots \\ \frac{e_{\max}}{\pi - 0.5 e_{\max}} & k = 2, 4, 6, \end{cases}$$

with  $\varepsilon > 0$ , then, even if each feedback gain  $K_k$  with  $k \in \mathbb{Z}$  is ISS, while the odd gains guarantee robust positive invariance, the even ones do not ensure it. Since any non-robust region  $X_k$  is surrounded by a robust one (see the second item of Theorem 5), we can state that global stability to the set of desired equilibrium points holds. Observe that, with appropriate approximations, if we limit the analysis to the attitude  $x \in [0, 2\pi]$ , such a scenario could be representative of the control of the yaw motion of a helicopter or of a single propeller ducted-fan where in one direction the energy of the control can be greater than in the opposite, because of the presence of the aerodynamic torque (this fact can be observed also in the heading dynamics – Eq. (25) – that will be considered in the application developed in Section 7). □

**Remark 4.** Please note that robust positive invariance is a peculiar property of discrete time systems. In [6], for the sake of avoiding chattering and ensure well-posedness, such a property has been extended to continuous time systems by means of classical “sample-and-hold” machineries. Formally the use of those kind of input holding techniques, often referred to in the literature as sample-and-hold (see [7]), makes the overall control law to become

$$u(k) = \begin{cases} g_1\left(y\left(\left\lfloor \frac{t}{T_s} \right\rfloor T_s\right)\right), & \text{if } y\left(\left\lfloor \frac{t}{T_s} \right\rfloor T_s\right) \in X_1, \\ g_2\left(y\left(\left\lfloor \frac{t}{T_s} \right\rfloor T_s\right)\right), & \text{if } y\left(\left\lfloor \frac{t}{T_s} \right\rfloor T_s\right) \in X_2, \end{cases} \quad (9)$$

where  $T_s \in \mathbb{R}_{>}$ . Once the overall sampled data system is obtained, conditions that ensure initial state dependent stability and stability to a set of equilibrium points can be obtained by taking advantage of the theory presented in this section combined with classical sample-data system arguments. It is worth remarking that this kind of solution to the stated problem is usually very conservative.

## 5. Alternative switching policies

In this section, we will present an overview of switching policies able to guarantee initial state dependent stability to a set of stable equilibrium points that are alternative to the “abrupt” switching law (4). The common idea of the approaches we will discuss is the use of control laws in which the scheduling policy does not depend only on the instantaneous value of the output. This is made by the introduction of (sometimes implicit) memory capabilities. In particular, in the first two subsections we will introduce strategies based on the hold-in-time ideas, while in the third the classical state-space-hysteresis is used. Finally in the last subsection a mixed strategy that combines the advantages of the presented approaches is proposed.

### 5.1. Alternative switching policies – Initial state dependent laws

A very simple way to deal with the problem of initial state dependent stabilization for both continuous and discrete-time systems is the introduction of control policies which are discontinuous only in the initial condition, i.e. the switching policy only depends on  $y(0)$ :

$$u(t) = \begin{cases} g_1(y(t)) & \text{if } y(0) \in X_1, \\ g_2(y(t)) & \text{if } y(0) \in X_2. \end{cases} \quad (10)$$

Because of its definition this kind of law is the natural solution to the initial state dependent stabilization problem, as is shown in the

following theorem:

**Theorem 6.** Let the system (1) and the piecewise continuous control law (10) be given. If  $g_i(y(t))$  is such that the closed-loop system  $\delta x(t) = f(x(t), g_i(y(t)))$  with equilibrium point  $\bar{x}_i$  is ISS with restriction  $S_i$  on the initial state, with  $X_i \subseteq X_i \oplus E \subseteq S_i$ , and  $\bar{x}_i \in X_i$  for  $i = 1, 2$ , then initial state dependent stability is guaranteed.

**Proof.** The proof follows from the ISS properties of the control laws and from the fact that the system never switches.  $\square$

The intuition behind such a law is that, at time  $t = 0$ , the equilibrium point to reach is decided.

**Example.** Let us consider our example, with the following control law

$$u_k(t) = -K(y(t) - 2k\pi), \quad \forall y(0) \in X_k$$

and with  $K = e_{\max}/(\pi - 0.5 e_{\max})$  the same state feedback gain that in the first example could yield to chattering, in this case such a law is able to perform initial state dependent stabilization since no switching of the control strategy is possible after the initial time  $t = 0$ .  $\square$

Even if these kinds of laws are the natural solution to the initial state dependent stabilization and they can be employed in a very simple way, it could be “not convenient” in many practical applications where stability to a set of points is enough because of the reduced flexibility it introduces. In order to show the limitations of (10), let us consider our usual attitude example in which we suppose that the state may be affected by a single impulsive disturbance  $w(t) = 0 \forall t \neq t'$  acting at time  $t'$

$$\begin{aligned} x(t+1) &= x(t) + u(t) + w(t), \\ y(t) &= x(t) + e(t). \end{aligned} \quad (11)$$

Assume that there is no measurement noise and that  $x(0) = y(0) = \pi$ . As a consequence the law  $u_0(t) = -K(y(t))$  is chosen and held for all time  $t \geq 0$ . Let us suppose now that the disturbance has the following expression:

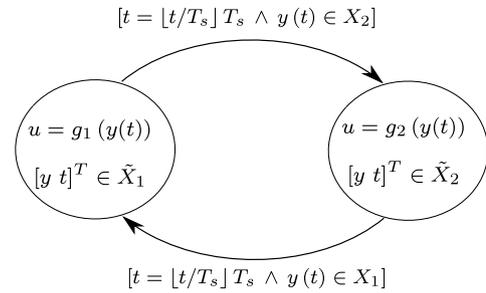
$$w(t) = \begin{cases} \pi + K\pi & \text{if } t = 1, \\ 0 & \text{otherwise.} \end{cases}$$

This yields  $x(1) = y(1) = 2\pi$ . It is worth noting that a point is one of the admissible equilibria for the system but, because of the control law (10), the closed loop system will be forced to converge to the origin. However, since the points 0 and  $2\pi$  correspond to the same orientation on the original configuration manifold  $S_1$ , the closed loop system will make a complete turn to recover the same orientation. This undesired behavior in the attitude control literature is known as the *unwinding phenomenon* (see for example [5]). This drawback of initial state dependent laws suggests to investigate more flexible strategies, in which the choice of the equilibrium to be reached asymptotically is not decided once and for all in a single instant of time.

**Remark 5.** It is very important to note that in order to ensure robustness of the stability to a set of points also with respect to this kind of process noise, the condition of ISS for each control law  $g_i(y(t))$  has to be extended to the entire domain  $X$ , and not only on the restriction  $S_i$ , with  $X_i \subseteq X_i \oplus E \subseteq S_i$ .  $\square$

## 5.2. Alternative switching policies – policy Holding

In Remark 4 we have presented a strategy introduced by [6] in which the control input is sampled and held for a certain amount



**Fig. 3.** The hybrid automaton which defines the policy holding control strategy.  $\tilde{X}_1 = (X \times (\mathbb{R}_{>} - Q)) \vee (X_1 \times Q)$ ,  $\tilde{X}_2 = (X \times (\mathbb{R}_{>} - Q)) \vee (X_2 \times Q)$ ,  $Q = \{q = \lfloor t/T_s \rfloor T_s \mid t \in \mathbb{R}_{>}\}$ .

of time  $T_s$ . The main disadvantage of such a strategy is that, for large values of  $T_s$ , it may give rise to a degradation of the dynamics because of the introduction of the sample-and-hold machinery in the control loop. Since the anomalous behavior results from the changes of strategy, a possible way to address the problem, overcoming the above performance limitation, is to sample and hold only the decision to switch between the control laws. Such a strategy can be formalized as follows:

$$u(t) = \begin{cases} g_1(y(t)) & \text{if } y(\lfloor t/T_s \rfloor T_s) \in X_1 \\ g_2(y(t)) & \text{if } y(\lfloor t/T_s \rfloor T_s) \in X_2 \end{cases} \quad (12)$$

where  $T_s \in \mathbb{N}$  for the discrete time case and  $T_s \in \mathbb{R}_{>}$  for the continuous-time case. The idea of this law is that while the “dynamic control laws” are continuous or have their own sampling time, the switching policy is governed by a slower sampling time ( $T_s$  times slower than the dynamic one in the discrete time case). This control scheme can be represented by means of the hybrid automaton in Fig. 3.

The first step to study the properties of this policy is to extend the definition of positive invariance given in Section 4 by considering explicitly the sample time of the control policy.

**Definition 7 (Robust Positive Invariance at Every Time  $T$ ).** Given system (1), the set  $X_p \subset X$  is said to be “robustly positive invariant at every time  $T$ ” with respect to a certain control law  $u(t) = g(y(t))$  if, for all initial conditions such that  $y(0) \in X_p$  the output  $y(kT)$  belongs to the set  $X_p$  for all  $k \in \mathbb{N}$ .  $\square$

The latter definition allows us, similarly to what has been done in Section 4, to state the following results:

**Theorem 8.** Let the system (1) and the piecewise continuous control law (12) be given. Moreover let  $g_i(y(t))$  be such that the closed-loop system  $\delta x(t) = f(x(t), g_i(y(t)))$  with equilibrium point  $\bar{x}_i$  is ISS with restriction  $S_i$  on the initial state, with  $X_i \subseteq X_i \oplus E \subseteq S_i$  and  $\bar{x}_i \in X_i$  for  $i = 1, 2$ . Then the following holds

- [PH1] initial state dependent stabilization is ensured if  $X_i$  is a robust invariant set at every time  $T_s$  with respect to  $g_i(y(t))$ ,  $i = 1, 2$ ; the latter is also a necessary condition if  $\varepsilon_1 \cap \varepsilon_2 = \emptyset$ ;
- [PH2] global set stabilization to the set of points  $\{\bar{x}_1, \bar{x}_2\}$  is ensured if  $X_1$  or  $X_2$  is a robust invariant set at every time  $T_s$  with respect to  $g_1(y(t))$  or  $g_2(y(t))$ , respectively.

**Proof.** Let us concentrate on the first item of the theorem, compactly denoted as [PH1]. Given a certain  $y(0) \in X_i$ , the control law  $g_i(y(t))$  is applied. From ISS, if the control never switches, the system asymptotically converges to a neighborhood of  $\bar{x}_i$ . Because of the invariance condition, for all  $t \geq T_s$ , the output  $y(\lfloor t/T_s \rfloor T_s) \notin X_j$  with  $j \neq i$ , then the control law never switches and then the system converges to a neighborhood of  $\bar{x}_i$  proving initial state dependent stabilization. In the case  $\varepsilon_1 \cap \varepsilon_2 = \emptyset$ , the necessity can

be proven by contradiction, following the same lines of the proof of the second item in [Theorem 5](#) in [Section 4](#).

Let us now focus on the last item, [PH2], of the theorem. Assume  $y(t) \in X_i$ . Because the applied control laws are ISS on some set  $S_i$ , with  $X_i \subseteq X_i \oplus E \subseteq S_i$ ,  $i = 1, 2$ , then, if the law is not robustly invariant, either the control law never switches (and hence  $y(t)$  converges to  $\bar{x}_i$ ), or there exists a finite time  $kT_s > 0$ , such that  $y(kT_s) < \infty$  and  $y(kT_s) \in X_j$ . In the latter case, the control law switches to  $g_j$ . Because  $X_j$  is invariant at every time  $T_s$  the control law will never switch again and  $y(k)$  will converge to  $\bar{x}_j$ .  $\square$

**Remark 6.** Note that (only for the discrete time case) the control law (12) can be seen as a generalization of the abrupt switching control law (4) discussed in [Section 2](#). Such a control law can in fact be recovered by setting  $T_s = 1$ . Moreover note that the strategy (12) is also a generalization of the initial state dependent laws (10) as  $T_s \rightarrow +\infty$ . Finally note that (12) can be generalized by considering different policy decision times  $T_{s,i}$ , one for each control law. In (12) such a generalization was avoided for the sake of clarity.  $\square$

It is of interest to study the behavior of the stated policy holding strategy in the case that an unexpected impulsive event instantaneously modifies the state values. Let us consider the system

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) + w(t) \\ y(t) &= x(t) + e(t) \end{aligned} \quad (13)$$

where  $w(t) \in \mathbb{R}^n$ ,  $w(t) = 0 \forall t \neq t'$  with  $t' > 0$ . As highlighted in [Section 5.1](#), if a non-proper policy is employed this kind of disturbance can make the system to behave in a non-convenient way, producing unwinding-like effects. The following result resumes the possible system's behavior in the case a policy holding strategy is employed:

**Lemma 9.** *Let us consider the system (13) and the control law (12). Then*

- (i) *if the conditions expressed in (PH1) hold, global stability to a set of points is guaranteed and the control policy switches at most once at time  $t_{s1} = \left\lfloor \frac{t'+T_s}{T_s} \right\rfloor T_s$ .*
- (ii) *if the conditions expressed in (PH2) hold, global stability to a set of points is guaranteed and the control policy switches at most 3 times.*

**Proof.** This can be proved by enumeration of the possible cases. Let us concentrate on the second item and suppose, without loss of generality, that  $X_i$  is invariant while  $X_j$  is not.

- (i)  $y(0) \in X_i$ , then until  $t_{s1} < \infty$  the law never switches. When the impulsive event occurs,  $y(t_{s1}) < \infty$  can be such that
  - (a)  $y(t_{s1}) \in X_i$ , then because of the invariance hypothesis the system never switches and converges to  $\bar{x}_i$  (total 0 switch)
  - (b)  $y(t_{s1}) \in X_j$  then, it can:
    - i. converge to  $\bar{x}_i$  switching another time (total 2 switches)
    - ii. converge to  $\bar{x}_j$  (total 1 switch)
- (ii)  $y(0) \in X_j$  then until  $t_{s1} < \infty$  it can:
  - (a) switch at a certain time  $t_{s0} \leq t'$ , in such a case, with one switch more we have reached the situation of (1), then we converge to  $\bar{x}_i$  after a maximum of 3 switches.
  - (b) It does not switch for all  $t \leq t'$  then at time  $t_{s1}$ ,  $y(t_{s1}) < \infty$  can be such that:
    - (i)  $y(t_{s1}) \in X_i$  then the control law switches once and converges to  $\bar{x}_i$  (1 switch)
    - (ii)  $y(t_{s1}) \in X_j$  then the control law can
      - (ii).1 switch once to converge to  $\bar{x}_i$  (1 switch)
      - (ii).2 converge to  $\bar{x}_j$  (0 switch)

The case (PH1) follows from the same considerations (if it switches once, because of the invariance hypothesis, it will converge).  $\square$

**Remark 7.** It is worth pointing out that during the “blind time” between  $t'$  and  $t_{s1} = \left\lfloor \frac{t'+T_s}{T_s} \right\rfloor T_s$ , the control could yield a trajectory that is not converging to the desired equilibrium. Obviously such behavior depends on the design parameter  $T_s$ : it is relevant for large values of  $T_s$  and disappears (only in the discrete time case) if  $T_s = 1$ . This is a good reason to look for the smallest  $T_s$  which is able to guarantee the desired invariance properties or for alternative strategies able to take, in those critical case, instantaneous decisions.  $\square$

To complete the discussion note that a sufficient condition for the invariance every time  $T$  is the so-called *robust positive invariance* of a set  $X_i$  after a time  $T$ :

**Definition 10** (*Robust Positive Invariance After Time T*). Given system (1), the set  $X_p \subset X$  is said to be “robustly positive invariant after time  $T$ ” with respect to a certain control law  $u(t) = g(y(t))$  if, for all initial conditions such that  $y(0) \in X_p$  the output  $y(t)$  belongs to the set  $X_p$  for all  $t \geq T$ .  $\square$

Such a notion of invariance is extremely useful from a practical viewpoint. In fact, even if more conservative than invariance every time  $T$ , it can be more easily handled and, as will be detailed in [Section 6](#), constructive algorithms for the selection of sets that are invariant after  $T_s$  can be obtained by means of standard ISS arguments. Interestingly enough, such an invariance property allows one to define a slightly different policy holding strategy in which the  $i$ -th policy is constrained to not switch again for at least a time  $T_{s,i}$  after a switch occurs. To formally describe such a strategy let us consider the finite state machine with state  $q \in \{1, 2\}$ , with the following initialization function

$$q(0) = i \quad \text{if } y(t) \in X_i$$

and forced transitions

$$q(t) = 2 \mapsto q(t) = 1 \quad \text{if } y(t) \in X_1 \wedge t \geq t_{switch} + T_{s,2}$$

$$q(t) = 1 \mapsto q(t) = 2 \quad \text{if } y(t) \in X_2 \wedge t \geq t_{switch} + T_{s,1}$$

where  $t_{switch}$  represents the last time in which the control law has switched and it is initially chosen equal to zero. Then the overall strategy can be described as follows

$$u = \begin{cases} g_1(y(t)) & \text{if } q(t) = 1 \\ g_2(y(t)) & \text{if } q(t) = 2. \end{cases} \quad (14)$$

Please note that such a strategy presents conceptual similarities with the *dwelling time* ideas (see for example [3]) introduced in the literature for the stability analysis of switching systems. Moreover it is worth remarking that the same results of [Theorem 8](#) and [Lemma 9](#) can be obtained by substituting the requirement “ $X_i$  is a robust invariant set at every time  $T_s$  with respect to  $g_i(y(t))$ ” with the condition “ $X_i$  is a robust invariant set after time  $T_{s,1}$  with respect to  $g_i(y(t))$ ” with the only noticeable difference being that the switching time  $t_{s1}$  becomes:

$$t_{s1} = \begin{cases} t_{switch} + T_{s,q(t_{switch})} & \text{if } t' < t_{switch} + T_{s,q(t_{switch})} \\ t'^+ & \text{else} \end{cases} \quad (15)$$

where  $t'^+ = t'$  in the continuous time case, and  $t'^+ = t' + 1$  in the discrete time case. Note that the latter implies that the switching is instantaneous if  $t \geq t_{switch} + T_{s,q(t_{switch})}$  where  $q(t_{switch})$  denotes the feedback law applied after the last switching event. This enhanced flexibility is due to the fact that the “blind time” in which the strategy cannot change policy reduces to the single temporal window  $T_{s,i}$  after a switch into the  $i$ -th region.

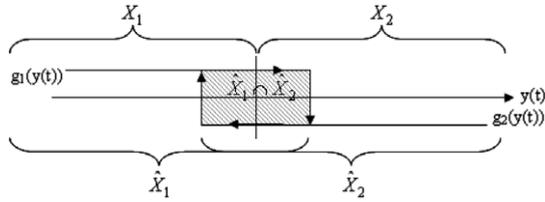


Fig. 4. Hysteresis on the state space.

**Example.** Let us consider the attitude control example and assume that the measurement noise is such that  $|e(t)| \leq e_{\max} < \pi/2$  for all  $t$  positive. For any  $y(0) \in X_k$ , we look for the value of the control parameter  $K$  such that the set  $X_k$  is robustly positive invariant after time  $T_s$  that means we are looking for a value of  $K$  such that the corresponding state of the system satisfies the constraint  $x(t + T_s) = x(t) + u(t) + u(t + 1) + \dots + u(t + T_s) \in X_k^*(e_{\max})$  for any  $x(t) \in X_k^*(-e_{\max})$ . Simple calculations show that robust positive invariance after a time  $T_s$  is obtained if  $K > 2e_{\max}/(T_s(\pi - 2e_{\max}))$ . Note that, because of the last relationship, for an increased  $T_s$ , a small value of  $K$  is allowed.  $\square$

**Remark 8.** The above example suggests that policy holding strategies can be used to simplify the overall design of the control law, decoupling the synthesis of the low level feedback law from the tuning of the switching policy: in a first stage it is possible to choose the parameters of the control feedback (almost) ignoring the effects of the measurement noise, and then, in a second stage, to choose the smallest time  $T_s$  that guarantees the needed invariance properties.  $\square$

### 5.3. Alternative switching policies – Hysteresis

As is well known, a classical way to deal with many of the problems arising from the use of switching control laws is the introduction of hysteresis regions (see for example [24]). In the continuous time framework such a kind of solution has been adopted to deal with the chattering problem in [8]. Here we will characterize the properties of this kind of strategy when applied to the problem under analysis underlining the advantages and disadvantages with respect to the previously described policies. In order to formally define the hysteresis strategy consider the finite state machine with state  $q \in \{1, 2\}$ , with the following initialization function

$$q(0) = i \quad \text{if } y(t) \in X_i$$

and forced transitions

$$q(t) = 2 \mapsto q(t) = 1 \quad \text{if } y(t) \in \bar{X}_1$$

$$q(t) = 1 \mapsto q(t) = 2 \quad \text{if } y(t) \in \bar{X}_2$$

where, as depicted in Fig. 4,  $\bar{X}_1 = X - \hat{X}_2$ ,  $\bar{X}_2 = X - \hat{X}_1$  are the “immediate switching” regions,  $\hat{X}_1 \supset X_1$  complying with  $\hat{X}_2 \supset X_2$  and  $\hat{X}_1 \cap \hat{X}_2$  is the hysteresis region. By using the above finite state machine we can now introduce the following hysteresis-based control law

$$u = \begin{cases} g_1(y(t)) & \text{if } q(t) = 1 \\ g_2(y(t)) & \text{if } q(t) = 2. \end{cases} \quad (16)$$

The idea behind this law is that, when the output  $y(t)$  belongs to the hysteresis region, a switching of the control law is always forbidden with the only exception of the initial time. To analyze the hysteresis control laws we need to introduce a new definition of positive invariance:

**Definition 11** (Robust Positive Invariance with Initial Output in  $X_0$ ). Given a system (1) and a certain control law  $u(t) = g(y(t))$ , the set  $X \supseteq X_0$  is said to be “robustly positively invariant with initial output in  $X_0$ ” with respect to a control law  $u(t) = g(y(t))$  if, for any initial output  $y(0) \in X_0$ , the output evolution always belongs to  $X$  in forward time i.e.

$$y(t) \in X, \quad \forall y(0) \in X_0, \quad \forall t \geq 0. \quad \square$$

In a similar way to what has been done in the previous subsections, it is possible to state, by using the definition of robust invariance, the following results about the initial state dependent stabilization and the global stability to a set of points.

**Theorem 12.** Let the system (1) and the piecewise continuous control law (16) be given. Moreover let  $g_i(y(t))$  be such that the closed-loop system  $\delta x(t) = f(x(t), g_i(y(t)))$  with equilibrium point  $\bar{x}_i$  is ISS with restriction  $S_i$  on the initial state, with  $\hat{X}_i \subseteq \hat{X}_i \oplus E \subseteq S_i$ , and  $\bar{x}_i \in X_i$  for  $i = 1, 2$ . Then the following holds

- [PH1'] initial state dependent stabilization is ensured if  $\hat{X}_i$  is a robust invariant set with initial output in  $X_i$  with respect to  $g_i(y(t))$ ,  $i = 1, 2$ ; the latter is also a necessary condition if  $\varepsilon_1 \cap \varepsilon_2 = \emptyset$ ;
- [PH2'] global set stabilization to the set of points  $\{\bar{x}_1, \bar{x}_2\}$  is ensured if  $\hat{X}_1$  or  $\hat{X}_2$  is a robust invariant set with initial output respectively in  $X_1$  or  $X_2$  with respect to  $g_1(y(t))$  or  $g_2(y(t))$ .

**Proof.** The first item, [PH1'], of the theorem can be proved as follows. Because of the invariance condition, the control law never switches, then, because the closed loop system is ISS  $\forall y(0) \in X_i$ , initial state dependent stabilization is ensured. Let us consider the case in which  $\varepsilon_1 \cap \varepsilon_2 = \emptyset$ . Let suppose there exists a certain  $y(0) \in X_i$  such that the trajectory converges asymptotically to a neighborhood of  $\bar{x}_i$  and at a certain time  $y(k)$  is such that  $y(k) = y' \in \hat{X}_j$ . Let now state a certain  $y(0) = y' \in X_j$  should converge to a neighbor of  $\bar{x}_j$ . If the neighbors of  $\bar{x}_i$  and  $\bar{x}_j$  are disjointed this proves the necessity by contradiction.

The last item, [PH2'], of the theorem can be proved following the same arguments adopted in the proof of the second item of Theorem 5 in Section 4  $\square$

We complete here the analysis of the hysteresis strategy by considering the case in which an impulsive event perturbs the state of the system at a certain time  $t'$ .

**Lemma 13.** Let us consider the system (13) and the control law (16). Then

- (i) if conditions expressed in [PH1'] hold, global stability to a set of points is guaranteed and the control policy switches at most once;
- (ii) if conditions expressed in [PH2'] hold, global stability to a set of points is guaranteed and the control policy switches at most 3 times.

**Proof.** The proof follows from enumeration of the possible cases, similarly to what has been done in Lemma 9. The only relevant difference can be shown proving the first item. Assume that  $y(0) \in X_i$ , then

- (i) if  $y(t') \in \bar{X}_j$  ( $y(t' + 1) \in \bar{X}_j$  for the discrete case), the law switches instantaneously to  $g_j$ , and the state converges to a neighborhood of  $\bar{x}_j$ ;
- (ii) if  $y(t') \in X_i$  ( $y(t' + 1) \in X_i$  for the discrete time case) the law never switches and converges to a neighborhood of  $\bar{x}_i$ ;
- (iii) if  $y(t') \in \hat{X}_i - X_i$  ( $y(t' + 1) \in \hat{X}_i - X_i$  for the discrete time case) the law can converge to a neighborhood of  $\bar{x}_i$  without any switch or to a neighborhood of  $\bar{x}_j$  switching at most once, at a certain time  $t > t'$ .  $\square$

**Remark 9.** Notice that the hysteresis policy is able to perform an instantaneous change of policy for an impulsive disturbance such that  $y(t) \in \bar{X}_j$ . Otherwise, even if initial state stability conditions hold true, if  $y(t) \in \hat{X}_i - X_i$ , then a switch may occur at a certain  $t > t'$ . Note that, in general, this time could be arbitrarily large depending on the properties of the closed loop system, while in the case of the policy holding discussed in the previous subsection in Lemma 9, it has been shown to be upper-bounded by  $t_{s1}$ . □

It is interesting to note a certain conceptual similarity between hysteresis policies and holding policies introduced in the previous subsection: while the first is a “holding” condition which is performed in the state space, the second can be seen as a sort of “hysteresis with respect to time”.

5.4. Alternative switching policies – Mixed strategy holding/hysteresis

In the two previous subsections we have introduced switching policies capable of ensuring initial state dependent stabilization and global stabilization to a set of points for both continuous and discrete time systems. In the case of instantaneous modification of system state, which may happen because of the presence of an impulsive disturbance, we have shown that both strategies present some disadvantages. While hysteresis has revealed capable to immediately change the control law if the state of the system is outside the “hysteresis region” and then closed to a desired equilibrium point, only holding strategies can guarantee a fixed upper-bound to the maximum time before a switch. This fact suggests to look for “mixed” strategies in order to take advantage of both the above properties at the same time. The idea is to consider a policy holding strategy (in this case the one based on the invariance after  $T_s$ ) and to compute immediate switch areas in which the system can immediately change control laws being sure that it is a “safe switch” and can be expressed by the following policy

$$u = \begin{cases} g_1(y(t)) & \text{if } q(t) = 1 \\ g_2(y(t)) & \text{if } q(t) = 2. \end{cases} \quad (17)$$

where  $q(t) \in \{1, 2\}$  is the state of the finite state machine with the following initialization function

$$q(0) = i \quad \text{if } y(t) \in X_i$$

and forced transitions

$$q(t) = 2 \mapsto q(t) = 1$$

$$\text{if } (y(t) \in X_1 \wedge t \geq t_{switch} + T_{s,2}) \vee (y(t) \in \bar{X}_1)$$

$$q(t) = 1 \mapsto q(t) = 2$$

$$\text{if } (y(t) \in X_2 \wedge t \geq t_{switch} + T_{s,1}) \vee (y(t) \in \bar{X}_2)$$

where  $t_{switch}$  represents the last instant time the control law switched which is initially chosen equal to zero. By construction we assume that the sets  $X - \bar{X}_i$ ,  $i = 1, 2$ , are robustly positive invariant with initial output in  $X_j$ , for  $i \neq j$ . Under this hypothesis properties of system (1) equipped with the law (17) are determined only by the robust positive invariance after time  $T_{s,i}$  of the sets  $X_i$  for  $i = 1, 2$ , according to the results given in Theorem 8 for the policy holding strategies.

The case of an impulsive disturbance is then of interest and it is summarized by the following lemma.

**Lemma 14.** Let us consider the system (13) and the control law (17). Let the set  $\bar{X}_i$  be such that  $X - \bar{X}_i$  is robustly positive invariant with initial output in  $X_j$ , for  $i \neq j$ ,  $i = 1, 2$ . Then

- (i) if  $X_i$  is a robust invariant set after a time  $T_{s,i}$  with respect to  $g_i(y(t))$ ,  $i = 1, 2$ , then global stability to a set of points is guaranteed and the control policy switches at most once at time  $t_s \leq t_{s1}$ , with  $t_{s1}$  defined in (15);

- (ii) if  $X_1$  is a robust invariant set after a time  $T_{s,1}$  or  $X_2$  is a robust invariant set after a time  $T_{s,2}$  with respect to respectively  $g_1(y(t))$  or  $g_2(y(t))$ , then global stability to a set of points is guaranteed and the control policy switches at most 3 times.

**Proof.** This can be proven by enumeration of the possible cases, following the same line of the proof of Lemma 9. The main difference in the proof can be shown proving the first item. Assume that  $y(0) \in X_i$ , then

- (i) if  $y(t') \in X_i$  (respectively  $y(t'+1) \in X_i$  for the discrete time case) the law never switches and the state converges to a neighborhood of  $\bar{x}_i$ ;
- (ii) if  $y(t') \in \bar{X}_j$  (respectively  $y(t'+1) \in \bar{X}_j$  for the discrete time case) the law switches instantaneously to  $g_j$ , and the state converges to a neighborhood of  $\bar{x}_j$ ;
- (iii) if  $y(t') \in X_j - \bar{X}_j$  and if there exists a time  $t_{hs}$ , with  $t' < t_{hs} < t_{s1}$  such that  $y(t_{hs}) \in \bar{X}_j$  the law switches to  $g_j$ , and the state converges to a neighborhood of  $\bar{x}_j$ , otherwise the law to be applied will be decided at time  $t = t_{s1}$  and in particular
  - (a) if  $y(t_{s1}) \in X_i$ , then the law never switches and the state converges to  $\bar{x}_i$ ;
  - (b) if  $y(t_{s1}) \in X_j$ , then the law switches once at time  $t_{s1}$  and the state will converges to  $\bar{x}_j$ . □

**Remark 10.** The rationale of the above Lemma is that the presented mixed strategy introduces some degree of flexibility w.r.t. policy holding and hysteresis strategies. The main reason is that it allows us to opportunistically change the actual law even before the prescribed “holding time”  $T_{s,i}$ ,  $i = 1, 2$  in all those situations in which the invariance properties ensure that it can be done safely.

6. Building robust positive invariant sets from ISS-Lyapunov functions

The goal of this section is to show how the ISS property of the feedback laws can be employed to build robust positive invariant sets satisfying the Definitions 4, 10 and 11. The idea is to take advantage of the fact that ISS implies that there exists an ISS-Lyapunov function which can be used to build forward invariant sets in both the state and output space.

For the sake of conciseness we focus our attention directly on the closed-loop dynamics obtained from system (1) once a certain law  $g_i(y(t))$  in (4) is selected:

$$\begin{aligned} \delta \tilde{x}(t) &= f_c(\tilde{x}(t), e(t)) \\ \tilde{y}(t) &= \tilde{x}(t) + e(t) \end{aligned} \quad (18)$$

where  $\tilde{x}(t) := x(t) - \bar{x}_i$ ,  $\tilde{y}(t) := y(t) + \bar{x}_i$  and  $f_c(\tilde{x}(t), e(t)) := f(\tilde{x}(t) + \bar{x}_i, g_i(\tilde{x}(t) + \bar{x}_i + e(t)))$  with  $f_c(0, 0) = 0$ . The system (18) is ISS with respect to the exogenous input  $e$  if and only if – see among other [22,10] – there exist class  $K$  functions  $\alpha, \bar{\alpha}, \chi$  and a function  $V(\tilde{x}) : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq}$ , denoted as an ISS-Lyapunov function, such that for all  $\tilde{x} \in \mathbb{R}^n$

$$\alpha(\|\tilde{x}\|) \leq V(\tilde{x}) \leq \bar{\alpha}(\|\tilde{x}\|) \quad (19)$$

and for all  $\tilde{x}$  such that  $\|\tilde{x}\| \geq \chi(e_{max})$

$$\frac{\partial V}{\partial \tilde{x}} f_c(\tilde{x}, e) \leq -\alpha(\|\tilde{x}\|) \quad (20)$$

for continuous time systems and

$$V(f_c(\tilde{x}, e)) - V(\tilde{x}) \leq -\alpha(\|\tilde{x}\|) \quad (21)$$

for discrete time systems.

The definition of an ISS-Lyapunov function given above allows one to obtain some invariance properties on the state trajectories of the closed-loop system. Along this line it is possible to prove (see e.g. [22]) the following lemma:



Fig. 5. The prototype of miniature ducted-fan aerial vehicle described in [28].

**Lemma 15.** Let us consider system (18) and assume that an ISS-Lyapunov function satisfying (19) and (20) (or respectively (21) for the discrete time case) is given. Let  $c^* := \bar{\alpha} \circ \chi(e_{\max})$  and let  $c$  be a positive real number such that  $c > c^*$ . Assume that  $\tilde{x}(0) \in B_{\bar{\alpha}^{-1}(c)}$ .<sup>2</sup> Then the following holds:

- $\tilde{x}(t) \in B_{\bar{\alpha}^{-1}(c)}$  for all  $t \geq 0$ ;
- for all  $c'$  such that  $c > c' > c^*$  there exists a time  $T_{c \rightarrow c'} > 0$  such that for all  $t \geq T_{c \rightarrow c'}$  then  $x(t) \in B_{\bar{\alpha}^{-1}(c')}$ .

It is worth noting that the time  $T_{c \rightarrow c'} > 0$  can be upper-bounded by a positive value  $\bar{T}_{c \rightarrow c'}$  which can be obtained as the solution of the boundary value problem for the differential equation  $\dot{V}(t) = -\alpha \circ \bar{\alpha}^{-1}(V(t))$  (or, for the discrete time case, for the scalar difference equation  $V(t+1) = V(t) - \alpha \circ \bar{\alpha}^{-1}(V(t))$ ) with the boundary conditions  $V(0) = c$  and  $V(\bar{T}_{c \rightarrow c'}) \leq c'$ .

Observe also that from the definition of the output  $\tilde{y}(t)$  and from the assumptions stated in Section 2 for the measurement noise  $e(t)$ , for all  $c > 0$  it holds:

$$\tilde{x} \in B_c \Rightarrow \tilde{y} \in B_{c+e_{\max}}, \quad \tilde{y} \in B_c \Rightarrow \tilde{x} \in B_{c+e_{\max}}. \quad (22)$$

Then, taking advantage of Lemma 15 and of (22) it is possible to directly build for the system (18) the sets which satisfy the Definitions 10 and 11 according to the two following lemmas.

**Lemma 16.** Let us consider system (18) and assume that an ISS-Lyapunov function satisfying (19) and (20) (or respectively (21) for the discrete time case) is given. Let (22) hold for all  $c > 0$  and let there be  $c^*$  defined in Lemma 15. Then the following holds:

- A set  $Y_0 \subseteq \mathbb{R}^n$ , such that  $\tilde{y}(0) \in Y_0$ , is a robust positive invariant set after a time  $T$  if
  - (i) there exists  $c, c'$  positive numbers with  $c > c' > c^*$  such that  $B_{\bar{\alpha}^{-1}(c')+e_{\max}} \subseteq Y_0 \subseteq B_{\bar{\alpha}^{-1}(c)-e_{\max}}$ ; (23)
  - (ii) the time  $T$  is larger than or equal to the time  $T_{c \rightarrow c'}$  defined in Lemma 15.
- A set  $\bar{Y}_0 \subseteq \mathbb{R}^n$  is a robust positive invariant set with initial output in  $Y_0$ , where  $Y_0 \subset \bar{Y}_0$  if there exists a positive  $c > c^*$  such that

$$Y_0 \subseteq B_{\bar{\alpha}^{-1}(c)-e_{\max}} \quad \text{and} \quad B_{\bar{\alpha}^{-1}(c)+e_{\max}} \subseteq \bar{Y}_0. \quad (24)$$

**Remark 11.** It is easy to prove that the same arguments hold also in the case of ISS with restriction  $S$  on the set of initial conditions provided that the restrictions are sufficiently large in order to have  $B_{\bar{\alpha}^{-1}(c)} \subseteq S$ . □

**Remark 12.** It is worth remarking that the above construction relies on the knowledge of an ISS-Lyapunov function for the closed-loop dynamics. For a large class of systems, an ISS-Lyapunov function may be determined by following the theory detailed for example in [22,25]. In order to make clear how the above Lemmas can be exploited to build the invariant sets we are interested in, please refer to Section 7 where a realistic case study is detailed. Finally note that less conservative results may be obtained by using the Lyapunov function  $V(\cdot)$  itself instead of its bounds  $\bar{\alpha}(\cdot)$  and  $\underline{\alpha}(\cdot)$ . However here we preferred to stress the fact that it is possible to construct invariant sets even in the (very common) case the analytical expression of  $V(\cdot)$  is too hard to be reasonably managed. □

## 7. An application: heading control of ducted-fan aerial vehicle

In this section we apply the switching policies developed in the paper in order to control the heading angle of a ducted-fan aerial vehicle – see among others [26,27]. In particular the prototype which will be considered here is described precisely in [28], towards which the reader is referred for further details about the mechanical structure and the electronic hardware employed to design the autopilot. The aircraft is composed by a fixed pitch propeller driven by an electric motor and a set of control vanes which are in charge of controlling the attitude dynamics by deviating the airflow generated by the propeller itself. By denoting with  $w_p$  the angular speed of the propeller, the velocity of the air inside the duct can be approximated – according to Froude theory (see among others [29]) – as

$$V_i = \sqrt{\frac{k_T w_p^2}{2\rho S_{\text{disk}}}}$$

where  $\rho$  denotes the air density,  $S_{\text{disk}}$  the area of the propeller disk and  $k_T$  a constant coefficient. Clearly  $V_i$  represents the relative wind which is used in the computation of the control torque. In fact, by denoting with  $u$  the angle of attack of the control vanes, it turns out that the torque generated by the flaps can be written as  $N_c = k_c w_p^2 u$  while the resistant aerodynamic torque of the propeller can be approximated by  $N_p = -k_D w_p^2$ , with  $k_c$  and  $k_D$  constant positive coefficients. Denoting with  $x_1$  the heading angle and with  $x_2$  the heading angular velocity, the heading dynamics can then be written as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ I_z \dot{x}_2 &= k_c w_p^2 u - k_D w_p^2 \end{aligned} \quad (25)$$

where  $I_z$  denotes the inertia around the vehicle vertical axis. The design of the heading control law for the ducted-fan consists of the choice of the input  $u$  in (25) in order to stabilize a given (constant) heading reference  $\bar{x}_1$ . Observe that in order for (25) to be controllable a necessary condition is that  $w_p > 0$  (see Fig. 5).

<sup>1</sup> With the notation  $\circ$  we denote the standard composition of functions, i.e. given two functions  $f(\cdot), g(\cdot)$ ,  $f \circ g = f(g(\cdot))$ .

<sup>2</sup>  $B_r$  denotes the ball of radius  $r$  in  $\mathbb{R}^n$  centered in the origin, i.e.  $x : \|x\| \leq r$ .

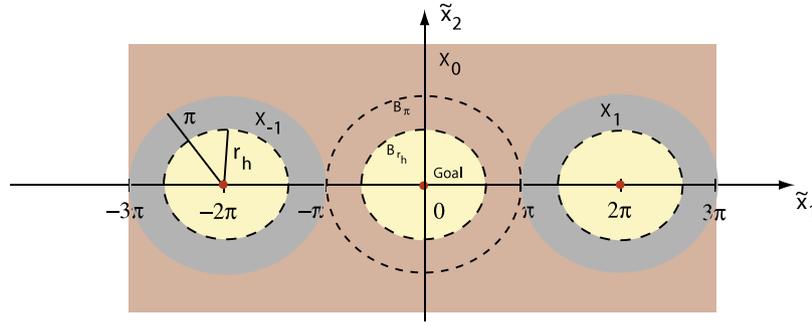


Fig. 6. The construction of the significative invariant sets for the problem of global-set stabilization of the heading angle of the ducted fan.

7.1. Design of the feedback control law

For the purpose of designing a feedback control law we rewrite the system (25) as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= a(t)(k_c u - k_D) \end{aligned} \quad (26)$$

with  $a(t) := w_p^2(t)/I_z$ . Let us assume that  $a(t) > 0$  for all  $t \geq 0$ , which in practice requires the propeller angular velocity to be greater than zero, as happens during a standard flight. Moreover we assume that the outputs available for feedback are given by

$$\begin{aligned} y_1 &= x_1 + e_1 \\ y_2 &= x_2 + e_2 \end{aligned} \quad (27)$$

where  $\|e_1(t)\|_\infty \leq e_{\max}^1$  and  $\|e_2(t)\|_\infty \leq e_{\max}^2$  are bounded measurement errors. For the real prototype an estimation of  $e_{\max}^1$  and  $e_{\max}^2$  has been obtained through a covariance analysis of the AHRS (Attitude and Heading Reference System) algorithm which is employed to reconstruct the attitude of the system from the available measures of an inertial measurement unit (IMU), which consists of an array of accelerometers and gyros, and magnetometers – see for more details [30].

The goal of the control law to be designed is to stabilize a certain constant heading orientation, i.e. to obtain global set-stabilization the set of points

$$\{x_1^* + 2k\pi\}, \quad k \in \mathbb{Z}$$

which represent the same orientation. In the following analysis the heading reference signal  $x_1^*$  will be allowed to be a piecewise constant function of time in order to define the desired orientation for the system. Let us introduce the following change of coordinates:

$$\begin{aligned} \tilde{x}_1^k &:= x_1 - (x_1^* + 2k\pi), \quad k \in \mathbb{Z} \\ \tilde{x}_2 &:= x_2 \\ \tilde{y}_1 &:= \tilde{x}_1^k + e_1 \\ \tilde{y}_2 &:= \tilde{x}_2 + e_2. \end{aligned}$$

In the new coordinates, system (26) can be rewritten as:

$$\begin{aligned} \dot{\tilde{x}}_1^k &= \tilde{x}_2 \\ \dot{\tilde{x}}_2 &= a(t)(k_c u - k_D). \end{aligned} \quad (28)$$

Let us consider the following choice of the control input  $u$

$$u = \frac{1}{k_c a(t)}(k_D a(t) + u')$$

where the remaining input  $u'$  is chosen as a PD control law

$$u' = -K_P(\tilde{y}_1 + K_D \tilde{y}_2)$$

with  $K_P > 0$  and  $K_D > 0$  control parameters. Due to the above choice of the control input, the closed loop error dynamics are given by the linear system:

$$\dot{x} = Ax + Be \quad (29)$$

where

$$\begin{aligned} x &= \begin{bmatrix} \tilde{x}_1^k \\ \tilde{x}_2 \end{bmatrix} & e &= \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} & A &= \begin{bmatrix} 0 & 1 \\ -K_P & -K_P K_D \end{bmatrix} \\ B &= \begin{bmatrix} 0 & 0 \\ -K_P & -K_P K_D \end{bmatrix}. \end{aligned}$$

We are now interested in building an ISS-Lyapunov function for the above linear system so that, with the help of the theory presented in Section 6, we can build the positive invariant regions required for the application of the switching policies presented in this paper. Let us fix  $K_P$  and  $K_D$  such that the matrix  $A$  is Hurwitz. Then there exists a symmetric and positive definite matrix  $P$  such that

$$PA + A^T P = -I.$$

Let be  $V(x) = x^T P x$  a candidate ISS-Lyapunov function. It holds that

$$\underline{a}(\|x\|)^2 \leq V(x) \leq \bar{a}(\|x\|)^2$$

where  $\underline{a}$  and  $\bar{a}$  can be chosen as the minimum and the maximum eigenvalue of  $P$ . Moreover

$$\frac{\partial V}{\partial x}(Ax + Be) \leq -\|x\|^2 + 2\|x\|\|P\|\|B\|\|e\|.$$

Let  $0 < \epsilon < 1$  and the function  $\chi : \mathbb{R}_\geq \rightarrow \mathbb{R}_\geq$  such that

$$\chi(u) := \frac{2}{1 - \epsilon} \|P\|\|B\|u$$

then for all  $x$  such that  $\|x\| \geq \chi(\|e\|_\infty)$  it holds

$$\frac{\partial V}{\partial x}(Ax + Be) \leq \alpha(\|x\|)$$

where  $\alpha : \mathbb{R}_\geq \mapsto \mathbb{R}_\geq$  is the linear function  $\alpha(u) = a u$  with

$$a := \frac{2\epsilon \|P\|\|B\|\|e\|_\infty}{1 - \epsilon}.$$

This fact proves that  $V(\cdot)$  is an ISS-Lyapunov function, since it satisfies conditions (19) and (20), for the error system (29).

7.2. Design of the switching regions

Let us concentrate on the state space of the error system (28). A possible choice of the partitions  $X_k, k \in \mathbb{Z}$ , containing the desired equilibrium points  $2k\pi$  has been depicted in Fig. 6. By using Lemma 16 we compute the time  $T$  such that the balls of radius  $\pi$  centered on the equilibrium points are robust positive invariants after a time  $T$ . This can be done by computing at first the two positive values  $c$  and  $c'$  such that (23) holds as an equality for the

**Table 1**  
System's parameters.

$I_z = 0.9 \times 10^{-2} \text{ kg m}^2$	$I_{prop} = 0.3 \times 10^{-4} \text{ kg m}^2$	$\rho = 1.2 \text{ kg/m}^3$
$k_c = 4.3 \times 10^{-4}$	$k_D = 12 \times 10^{-4}$	

choice  $Y_0 \equiv B_\pi$ , obtaining  $c = (\pi + e_{\max})\bar{a}$  and  $c' = (\pi - e_{\max})\underline{a}$ . As stated in Lemma 16, the two computed values of  $c$  and  $c'$  are required to be greater than a positive value  $c^*$ . The latter, for the above ISS-Lyapunov function, is

$$c^* = \frac{2\bar{a}e_{\max}}{(1 - \epsilon)} \|P\| \|B\|$$

and indeed a proper choice of the parameter  $\epsilon$  can be made in order to satisfy all the assumptions of the Lemma. Following Section 6 arguments, the time  $T$  can be estimated by solving the boundary value problem associated with the scalar differential equation  $\dot{V} = -(a/\bar{a})V$  with  $V(0) = c$  and  $V(T) = c'$ , obtaining

$$T = \frac{\bar{a}}{a} \ln \frac{c}{c'}$$

Secondly we compute the radius  $r_h$  such that the set  $B_\pi$  is a robust positive invariant set with initial output in  $B_{r_h}$  (see Fig. 6). This can be done, following the second item of Lemma 16, solving Eqs. (24) as equalities by choosing  $Y_0 \equiv B_{r_h}$  and  $\bar{Y}_0 \equiv B_\pi$  and obtaining

$$r_h = (\underline{a}/\bar{a})(\pi - e_{\max}) - e_{\max}$$

Finally, to have global set stabilization to the desired set of points, for  $k \in Z$  and  $k \neq 0$  we define the regions  $X_k$  as the balls of radius  $\pi$  centered in points  $2k\pi$ , and for  $k = 0$  we consider the set obtained as the complement of the sum of all the previous sets, which by construction contains the origin. Fig. 6 shows precisely the construction considering  $k \in \{-1, 0, 1\}$ . Observe that the above construction has a clear physical meaning. For each  $k \neq 0$  the sets  $X_k$  in fact define the points which are closer or equal to the equilibrium  $2k\pi$  than to other possible equilibrium points. The idea is then to drive the control policy on the basis of the Euclidean distance between the current output and the desired goal by making the sets  $X_k$  invariant.

7.3. Simulations

We start by fixing the control parameters  $K_p = 1$  and  $K_D = 0.72$  so that the matrix  $A$  in (29) is Hurwitz, and then, computing the matrix  $P$  as specified in the previous subsection, we obtained  $\bar{a} = 2.1$  and  $\underline{a} = 1.0$ . For the specific hardware employed it holds that  $e_{\max}^1 \approx 0.1$  rad and  $e_{\max}^2 \approx 0.01$  rad/s, and indeed we have  $e_{\max} = \|[e_{\max}^1 \ e_{\max}^2]^T\| \approx 0.1$ , which denotes the maximum amplitude of the noise affecting system (29) (see Table 1).

By choosing  $\epsilon = 0.65$ , we obtain  $c = 6.8, c' = 3.2, c^* = 3.1$  and the function  $\alpha(u) = 0.96 u$ . With the above values we compute the time  $T$  and the radius  $r_h$  obtaining respectively  $T = 1.7$  s and  $r_h = 1.4$ . Those values are used to implement a mixed policy holding – hysteresis switching policy, in which, with an eye at Fig. 6, the balls of radius  $r_h$  – yellow – represent the immediate switching regions while the balls of radius  $\pi$  – light blue – represent the switching region with a holding time  $T$ .

The simulations have been depicted in Figs. 7, 8 and 9: the first two figures show the trajectory and the phase plot of the error while the last one depicts the heading and the reference signal in the original coordinates. At time  $t = 0$  s the initial conditions are such that the output are read inside the set  $X_{-1}$ , which is the ball of radius  $\pi$  centered in the point  $-2\pi$  (see Point 1 in the figures). Then the error system converges to the point  $-2\pi$  governed by a holding switching policy in which the time

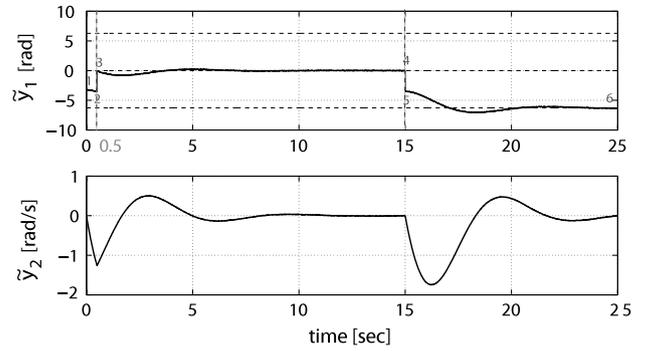


Fig. 7. The error output  $\tilde{y}_1(t)$  and  $\tilde{y}_2(t)$ .

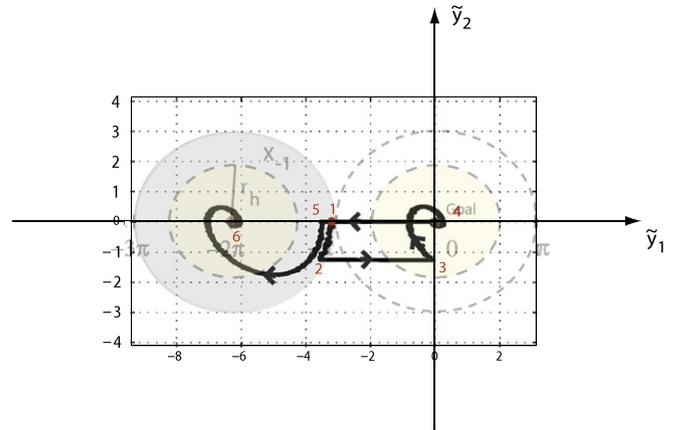


Fig. 8. The representation of the maneuver in the error coordinates  $(\tilde{y}_1-\tilde{y}_2)$ .

$T$  is fixed to 1.7 s. However at time  $t' = 0.5$  the reference is changed (Point 2 in the figures). This fact, in the error coordinates, is equivalent to an impulsive disturbance which causes the error system to reach a different state. Then (Point 3 in the figures) the output is read inside the circle of radius  $r_h$  centered in the origin, which is an immediate switching region. This causes the law to switch immediately, instead of waiting till the holding time is past, revealing the advantage of the opportunistic strategy in this latter situation. The error system then converges to the desired point until at time  $t = 15$  s the reference is changed again (Point 4 in the figures) and the output of the error system jumps inside the set  $X_{-1}$  (Point 5 in the figures). Then the control law is switched again to stabilize the point  $-2\pi$  (Point 6 in the figures).

8. Conclusion

This paper has dealt with a class of systems governed by piecewise continuous control laws subject to measurement noise in which more than one point in the state space could be a desired point to be stabilized. Through the sections we have described, both for discrete and continuous time systems, several switching policies able to guarantee that any possible state trajectory of the closed loop system asymptotically converges to one of the desired equilibria, avoiding undesired behaviors. After an analysis of the intrinsic “holding” nature of the discrete time framework, the paper has focused on the concepts of “strategy holding” in time and space, pointing out the invariance conditions needed to obtain the desired stability properties. Moreover the proposed strategies have been characterized in term of switching flexibility by studying the capability to react in a convenient way to sudden state changes. In this respect an innovative “opportunistic switching” policy capable of combining all the advantages of the previously presented strategies has been proposed. Finally synthesis aspects

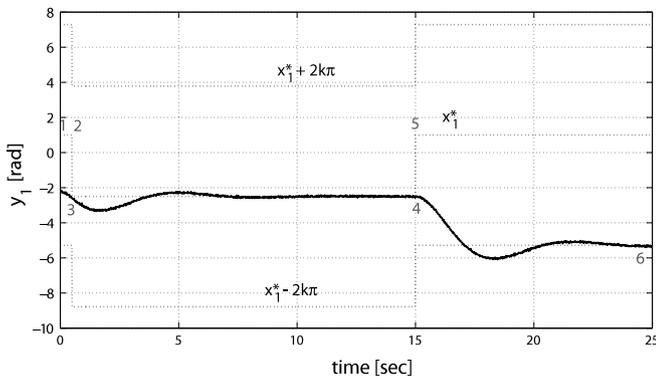


Fig. 9. The output  $y_1(t)$  and the set point  $x_1^*$ .

have been discussed and an applicative example based on the heading attitude control of a Ducted-Fan Aerial Vehicle has been developed to show the effectiveness of the presented strategies in a real application.

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