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Dilated model predictive control strategy for linear parameter-varying systems with a time-varying terminal set

G. Franzè[1] D. Famularo[2] E. Garone[1] A. Casavola[1]
[1]Dipartimento di Elettronica, Informatica e Sistemistica, Universitá degli Studi della Calabria, Via Pietro Bucci, Rende (CS), Italy
[2]DIMET, Facoltá di Ingegneria, Universitá degli Studi di Reggio Calabria, Reggio Calabria, Via Graziella, Italy
E-mail: franze@deis.unical.it

Abstract: We propose a model predictive control (MPC) strategy for input-saturated polytopic linear parameter-varying (LPV) discrete-time systems. Under the hypothesis that the plant parameter is measurable at each time instant, a dilation technique and parameter time-varying Lyapunov functions are exploited to reduce the conservativeness of pre-existing robust MPC paradigms. The contribution of this paper is to extend the MPC scheme presented in (Casavola *et al.* 2006) to the general case of control horizons of arbitrary length N, by considering the terminal constraint set to be time-varying. Feasibility and closed-loop stability of this strategy are proved and a final numerical example presented in order to show performance improvements with respect to pre-existing techniques.

1 Introduction

Model predictive control (MPC) is an optimisation-based control strategy capable to treat in an efficient manner the effect of constraints imposed on the normal plant operation. At each control interval the MPC algorithm computes an open-loop sequence of manipulated variable adjustements in order to optimise plant forecasts, compatibly with the prescribed constraints. The first input in the optimal sequence is injected into the plant, and the entire optimisation is repeated at subsequent time instants. Moreover, MPC is also known in literature to easily handle time-varying systems and to provide good tracking performance.

Linear parameter-varying (LPV) systems are by no means one of the frameworks where a receding horizon control strategy reveals its full ability. Strictly speaking a LPV system denotes a linear system whose elements in the state space representation are depending linearly or in an affine way from a set of time-varying parameters whose realisation is available at each time instant. Actually, this type of paradigm, like other MPC robust algorithms requires a relevant computational burden to achieve good regulation performance (see [1–6] and references therein).

Recently, starting from an idea of de Oliveira *et al.* [7] and Daafouz and Bernussou [8], more efficient MPC control laws have been derived by means of less conservative robust stability conditions than traditional quadratic stability approaches. The idea consists in the introduction of more flexible parameter-dependent Lyapunov functions (PDLFs) and in a matrix inequality relaxation result, also known as the *dilation trick* (see [9, 10]). Such a paradigm has been then extended to the case of LPV systems expressed via multi-model or affine representation in [11], by exploiting the assumption that the scheduling vector is available in real time.

Other notable approaches in literature to derive a less conservative approximation of the robust positive invariant region make use of terminal sets having a polyhedral structure (see [12–14] and references therein). This type of MPC algorithms, even if proposed for a robust multi-model uncertain plants framework can be easily translated into the LPV case. The major drawback of the polyhedral approach stands in the resulting optimisation procedure which exhibits a prohibitive combinatorial complexity in the control horizon length.

Here we develop a MPC control strategy for input saturated multi-model LPV systems based on dilation techniques. It will be shown that improved control performance are achievable with respect to traditional LPV receding-horizon control algorithms (see [1, 4, 15] and references therein).

We consider the general case of arbitrary N-step control horizons and assume the LPV scheduling vector measurable on-line at each time instant. A bank of state-feedback gains, computed off-line, is used as primal control law over which a sequence of N free control offsets, representing the degrees of freedom of the underlying optimisation problem, is superimposed. Then, the scheduling feedback gains are updated by means of a numerical procedure which takes advantage of the current state and the previously computed N free control inputs. It is shown that the proposed algorithm is solvable by means of a sequence of two semi-definite constrained programming problems involving linear matrix inequalities (LMIs). The resulting control law is applied to the plant in a receding horizon fashion and feasibility/closed-loop stability properties of the proposed strategy are summarised here. A numerical example is considered and comparisons with two strategies – a MPC LPV strategy based on an ellipsoidal approximation of the terminal set [15] and a RHC dilation based scheme [11] – are provided.

Notation: Given a matrix $\mathrm{X} \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$, we will denote with X_{ii}, $i = 1, \ldots, n$, the ith diagonal entry.

Given a symmetric matrix $\mathrm{P} = \mathrm{P}^{\mathrm{T}} \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$, the inequality $\mathrm{P} > 0$ $(\mathrm{P} \geq 0)$ denotes matrix positive definiteness (semi-definiteness). Given two symmetric matrices P, Q, the inequality $\mathrm{P} > \mathrm{Q}$ $(\mathrm{P} \geq \mathrm{Q})$ indicates that $\mathrm{P} - \mathrm{Q} > 0$, $(\mathrm{P} - \mathrm{Q} \geq 0)$.

Given a vector $x \in \mathbb{R}^{\mathrm{n}}$, the standard 2-norm is denoted by $\|x\|_2^2 = x^{\mathrm{T}}x$ whereas $\|x\|_{\mathrm{P}}^2 \triangleq x^{\mathrm{T}}\mathrm{P}x$ denotes the vector P-weighted two-norm.

The notation $\hat{v}_k(t) \triangleq v(t+k|t)$, $k \geq 0$ will be used to define the k-step ahead prediction of a generic system variable v from t onward under specified initial state and input scenario.

2 Problem statement

Let us consider the following discrete-time LPV system

$$\begin{cases} x(t+1) & = A(\theta(t))x(t) + B(\theta(t))u(t) \\ y(t) & = Cx(t) \end{cases} \tag{1}$$

with $x(t) \in \mathbb{R}^n$ denoting the state, $u(t) \in \mathbb{R}^m$ the input, $y(t) \in \mathbb{R}^p$ the output and all system matrices of compatible dimensions. The plant and input matrices are linear continuous functions of the scheduling parameter $\theta(t)$ having the following structure

$$A(\theta) = \sum_{j=1}^{l} \theta_j A_j, \quad B(\theta) = \sum_{j=1}^{l} \theta_j B_j \tag{2}$$

It is assumed that the range of the vector-valued parameter $\theta \triangleq [\theta_1, \theta_2, \ldots, \theta_l]^{\mathrm{T}}$ is limited to a compact subset $\Theta \subset \mathbb{R}^l$, with θ being the unit simplex

$$\Theta \triangleq \left\{ \sum_{j=1}^{l} \theta_j = 1, 0 \leq \theta_j \leq 1 \right\} \tag{3}$$

The following additional property is assumed for the family of plants (1)

- LPV - $\theta(t)$ is measurable at each time instant

We will denote as Θ^r and Θ^* the sets of all possible Θ-valued sequences of $r+1$ and, respectively, arbitrary length

$$\Theta^r \triangleq \left\{ \{\theta(t)\}_{t=0}^{r} : \theta(t) \in \Theta \right\}, \quad \Theta^* \triangleq \lim_{r\to\infty} \Theta^r \tag{4}$$

Moreover, the plant inputs are subject to componentwise saturation constraints of the type

$$|u_j(t)| \leq u_{j,\,\max},\ t \geq 0, \quad j = 1, \ldots, m \tag{5}$$

The aim is to find a regulation strategy $u(\cdot)$ which asymptotically stabilises (1)–(3) subject to (5). Among many, it is of interest to look for those strategies derived according to an optimality criterion which are able to guarantee a good dynamical behaviour to the regulated plant. A typical choice is the usual LQ index

$$J(x, u(\cdot)) \triangleq \max_{\theta(\cdot)\in\Theta^*} \sum_{t=0}^{\infty} \{\|x(t)\|_{\mathrm{R}_x}^2 + \|u(t)\|_{\mathrm{R}_u}^2\} \tag{6}$$

with $\mathrm{R}_u = \mathrm{R}_u^{\mathrm{T}} > 0$, $\mathrm{R}_x = \mathrm{R}_x^{\mathrm{T}} > 0$ suitable input and state weights.

3 Some previous results

A well known choice in literature is to restrict the control strategy to be a linear state feedback law

$$u(t) = \mathrm{F}x(t) \tag{7}$$

where $\mathrm{F} \in \mathbb{R}^{m\times n}$ denotes a gain matrix. Under this assumption, it is possible to derive a quadratic Lyapunov function

$$V(x(t)) = x^{\mathrm{T}}(t)\mathrm{P}x(t) \tag{8}$$

with $\mathrm{P} = \mathrm{P}^{\mathrm{T}} > 0$ such that the following result holds true

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Proposition 1: Given the multi-model uncertain plant (1) and let the couple (F, P) satisfy the set of Riccati inequalities

$$(A_j + B_j\mathrm{F})^{\mathrm{T}}\mathrm{P}(A_j + B_j\mathrm{F}) - \mathrm{P} + \mathrm{F}^{\mathrm{T}}\mathrm{R}_u\mathrm{F} + \mathrm{R}_x \le 0, \quad j = 1, \ldots, l \tag{9}$$

then, F robustly quadratically stabilises the plant family (1) and the following upper bound to the quadratic cost (6) holds true

$$J(x(0), \mathrm{F}x(\cdot)) \le x^{\mathrm{T}}(0)\mathrm{P}x(0) \triangleq \gamma, \quad \gamma > 0 \tag{10}$$

Moreover, the following ellipsoidal set

$$\mathcal{E}(\mathrm{P}, \gamma) \triangleq \{x \in \mathbb{R}^n | x^{\mathrm{T}}\mathrm{P}x \le \gamma, \gamma \ge 0\} \tag{11}$$

is a robust positively invariant region for the regulated state trajectory.

Proof: A proof is given in [16, 17]. □

On the basis of this result and by simple Schur transformations it is possible to recast in a receding horizon fashion the strategy (7) (see [18]). Recently, in order to ameliorate the control performance, the invariant set $\mathcal{E}(\mathrm{P}, \gamma)$ has been more flexibly defined by considering a general class of PDLFs, as proposed by [7]

$$V(x(t)) = x^{\mathrm{T}}(t)\mathrm{P}(\theta(t))x(t) \tag{12}$$

where

$$\mathrm{P}(\theta(t)) = \sum_{j=1}^{l} \theta_j(t)\mathrm{P}_j \tag{13}$$

Proposition 2 extends the results of Proposition 1 to the case of PDLFs.

Proposition 2: Let the $(l+1)$-tuple $(\mathrm{F}, \{\mathrm{P}_j\}_{j=1}^{l})$ satisfy the following set of l^2 Riccati inequalities (see [10])

$$(A_j + B_j\mathrm{F})^{\mathrm{T}}\mathrm{P}_k(A_j + B_j\mathrm{F}) - \mathrm{P}_j + \mathrm{R}_x + \mathrm{F}^{\mathrm{T}}\mathrm{R}_u\mathrm{F} \le 0, \quad j, k = 1, \ldots, l \tag{14}$$

Then, F robustly quadratically stabilises the plant family (1) and the following upper bound to the quadratic cost (6) holds true,

$$J(x(0), \mathrm{F}x(\cdot)) \le \max_{j=1,\ldots,l} x(0)^{\mathrm{T}}\mathrm{P}_j x(0) \triangleq \gamma, \; \gamma > 0 \tag{15}$$

Moreover, the vertices P_j, where $j = 1, \ldots, l$, are shaping matrices for the following family of ellipsoidal sets

$$\mathcal{E}(\mathrm{P}_j, \gamma) \triangleq \{x \in \mathbb{R}^n | x^{\mathrm{T}}\mathrm{P}_j x \le \gamma, \gamma \ge 0\}, \quad j = 1, \ldots, l \tag{16}$$

whose intersection

$$\bigcap_{j=1}^{l} \mathcal{E}(\mathrm{P}_j, \gamma) \tag{17}$$

is a robust positively invariant region for (1)–(3) under the state-feedback F.

Proof: See [10]. □

Exploiting this stabilisability result and using the same ideas of [18], a less conservative robust LMI-based MPC strategy was proposed in [9] and amended in [10]. It is worth to note that the control paradigms underlying Proposition 1 and Proposition 2, are especially tailored for multi-model uncertain systems because they do not make use of deriving $\theta(t)$ from measurements. Such a knowledge could potentially yield to more performing MPC strategies if the following scheduling control law

$$u(t) = \sum_{i=1}^{l} \theta_i(t)\mathrm{F}_i(t)x(t) \tag{18}$$

would be considered. In (18) $F_i(t)$, where $i = 1, \ldots, l$, denote a sequence of state-feedback gains computed on each system vertex (see [1, 15]). MPC strategies combining scheduling control laws (18) and PDLF have been proposed e.g. in [11] and their main results can be summarised in Proposition 3.

Proposition 3: Consider the linear time-varying system (1) under the LPV hypothesis and let $\mathrm{x}(t)$ be the measured state. Consider, at each time instant t, the following convex optimisation problem

$$\min_{\mathrm{Y,G,Q}_i} \gamma \tag{19}$$

subject to

$$\begin{bmatrix} \mathrm{G} + \mathrm{G}^{\mathrm{T}} - \left(\dfrac{\mathrm{Q}_i + \mathrm{Q}_j}{2}\right) & * & * & * \\ \left(\dfrac{\mathrm{A}_i\mathrm{G} + \mathrm{A}_j\mathrm{G} + \mathrm{B}_i\mathrm{Y}_j + \mathrm{B}_j\mathrm{Y}_i}{2}\right) & \mathrm{Q}_k & * & * \\ \mathrm{R}_x^{1/2}\mathrm{G} & 0 & \gamma\mathrm{I} & * \\ \mathrm{R}_u^{1/2}\left(\dfrac{\mathrm{Y}_i + \mathrm{Y}_j}{2}\right) & 0 & 0 & \gamma\mathrm{I} \end{bmatrix} \ge 0, \quad k = 1, \ldots, l, \; i = 1, \ldots, l, \; j = i, \ldots, l \tag{20}$$

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$$\begin{bmatrix} 1 & x^{\mathrm{T}}(t) \\ x(t) & \mathrm{Q}_j \end{bmatrix} \geq 0, \quad j = 1, \ldots, l \tag{21}$$

$$\begin{bmatrix} \mathrm{Z} & \mathrm{Y}_j \\ \mathrm{Y}_j^{\mathrm{T}} & \mathrm{G}^{\mathrm{T}} + \mathrm{G} - \mathrm{Q}_j \end{bmatrix} \geq 0, \quad j = 1, \ldots, l, \\ (Z)_{kk} \leq u_{k,\max}^2, \quad k = 1, \ldots, m, \tag{22}$$

If solvable, it allows one to define the following time-varying gain

$$\mathrm{F}(t) \triangleq \sum_{j=1}^{l} \hat{\theta}_j(t|t)\mathrm{F}_j(t) \tag{23}$$

$$\hat{\theta}_j(t|t) \triangleq \theta_j(t), \quad j = 1, \ldots, l \tag{24}$$

where $\mathrm{F}_j(t) = \mathrm{Y}_j(t)\mathrm{G}^{-1}(t)$, $j = 1, \ldots, l$ and $\mathrm{P}_j = \gamma \mathrm{Q}_j^{-1}$, $j = 1, \ldots, l$. Then, if a solution for the optimisation problem (19)–(22) there exists at the generic time instant t, it exists at each future time instants, the command satisfies the input constraints and the closed-loop system is asymptotically stable.

Proof: Reported in the Appendix. □

In order to improve the control performance, we will consider a generalisation of the above strategy represented by the following family of virtual commands

$$u(\cdot \mid t) = \begin{cases} u(t+k|t) = \sum_{i=1}^{l} \hat{\theta}_i(t+k|t) \\ F_i(t+k|t)x(t+k|t) \\ +c(t+k|t), & k = 0, 1, \ldots, N-1, \\ u(t+k|t) = \sum_{i=1}^{l} \\ \hat{\theta}_i(t+k|t)F_i(t+N|t)x(t+k|t), & k \geq N \end{cases} \tag{25}$$

where $\{\mathrm{F}_i(t+k|t)\}_{i=1}^{l}$, $k = 0, \ldots, N$ are suitable stabilising control laws and the sequence $c(\cdot|t)$ denotes N free perturbations over them. Given the set of stabilising gains $\{F_i(t+k|t)\}_{i=1}^{l}$, $k = 0, \ldots, N$, our strategy will consist of computing first the optimal sequence of control increments $c(\cdot|t)$ minimising an upper bound to the guaranteed cost and ensuring satisfaction of the prescribed input constraints and terminal set requirements. Then, an updating strategy for the terminal set will consist of computing $\{\mathrm{F}_i(t+N+1|t+1)\}_{i=1}^{l}$ and $\{\mathrm{P}_i(t+1)\}_{i=1}^{l}$ on-line and shifting backward the other control laws, viz.

$$\{\mathrm{F}_i(t+1+k|t+1)\}_{i=1}^{l} \leftarrow \{\mathrm{F}_i(t+k+1|t)\}_{i=1}^{l}, \\ k = 0, \ldots, N-1$$

The family of virtual commands (25) generalises the control structures of all previous MPC schemes. In particular for $N = 0$ one obtains the control structure used in [9] and [11] with $\mathrm{F}(t|t)$ updated on-line.

4 On-line MPC strategy and main result

In this section we will determine the conditions which guarantee the finiteness of the upper-bound to the quadratic cost (15) and ensure satisfaction of the prescribed constraints under the scheduling control law (25).

First of all, consider the convex hulls of k-step ahead state predictions starting from $x := x(t)$ at time t under the virtual command family (25). It results

$$\mathfrak{X}^{t|t} \triangleq \{x\} \tag{26}$$

$$\mathfrak{X}^{t+1|t} \triangleq \left\{ A_{\mathrm{F}}(\hat{\theta}(t|t))x + B(\hat{\theta}(t|t))\hat{c}(t|t) \right\} \tag{27}$$

$$\mathfrak{X}^{t+k|t} \triangleq \mathrm{co}\{A_{\mathrm{F}}(\hat{\theta}(t+k-1|t))z \\ + B(\hat{\theta}(t+k-1|t))\hat{c}(t+k-1|t), \\ \forall \hat{\theta}(t+k-1|t) \in \Theta^{k-1}(t),\ \forall z \in \mathrm{vert}\{\mathfrak{X}^{t+k-1|t}\}\} \tag{28}$$

where

$$\mathrm{A_F}(\hat{\theta}(t+k|t)) := \mathrm{A}(\hat{\theta}(t+k|t)) + \mathrm{B}(\hat{\theta}(t+k|t))\mathrm{F}(t+k|t)), \\ k \geq 0$$

According to (25), the cost (6) can be conveniently rewritten as follows

$$\mathrm{V}(x(t), \mathrm{P}_j(t), \mathrm{F}_j(\cdot\,|t), \hat{c}(\cdot\,|t)) := \\ \sum_{k=0}^{N-1} \left\{ \max_{z(k) \in \mathrm{vert}\{\mathfrak{X}^{t+k|t}(x(t))\}} \|z(k)\|_{\mathrm{R}_x}^2 + \|c(t+k|t)\|_{\mathrm{R}_u}^2 \right\} \\ + \max_{\substack{z \in \mathrm{vert}\{\mathfrak{X}^{t+N|t}(x(t))\} \\ \mathrm{P}_j(t), j = 1, \ldots, l}} \|z\|_{\mathrm{P}_j(t)}^2 \tag{29}$$

with $(\mathrm{P}_j(t) := \gamma(t)\mathrm{Q}_j^{-1}(t), \mathrm{F}_j(t) := \mathrm{Y}_j(t)\mathrm{G}^{-1}(t), \gamma(t))$, $j = 1, \ldots, l$, solution of the LMIs (20)–(22) with $x = x(t)$. Then, the overall algorithm is as follows

Algorithm: DL–LPV–TS

- *Initialisation*

Given $x(0)$ find

$$\left[\left\{ \mathrm{Q}_j(0), \mathrm{F}_j(\cdot\,|0)_{j=1}^{l}, \right\}_{j=1}^{l}, \mathrm{G}(0), \gamma(0) \right] := \arg \min_{Y_j, Q_j > 0, \gamma > 0} \gamma$$

subject to

* LMIs (20)–(22);
- $* \begin{bmatrix} 1 & z^{\mathrm{T}} \\ z & Q_j \end{bmatrix} \geq 0,\ j = 1, \dots, l,\ \forall z \in \mathrm{vert}\{\mathfrak{X}^{1|0}(x(t))\}$

Generic Step

1. For any $t \geq 0$, given $x(t)$, $\{P_j(t), F_j(t)\}_{j=1}^{l}$ and $\gamma(t)$, find

$$\hat{c}^*(\cdot\,|t) \triangleq \arg \min_{J_i>0,\hat{c}(\cdot|t)} \sum_{j=0}^{N} J_i \tag{30}$$

subject to

$$\begin{bmatrix} J_N & z' \\ z & \gamma^{-1}(t)Q_j(t) \end{bmatrix} \geq 0, \quad \forall z \in \ \mathrm{vert}\{\mathfrak{X}^{t+N|t}(x(t))\},\quad j = 1, \dots, l \tag{31}$$

$$J_N \leq \gamma(t) \tag{32}$$

$$|F(t)x(t) + \hat{c}(t|t)|_i \leq u_{i,\max}, \quad \forall i \in \{1, \dots, m\} \tag{33}$$

$$|F_j(t+k|t)z(k) + \hat{c}(t+k|t)|_i \leq u_{i,\max},\ \forall k \in \{1, \dots, N-1\}, \forall i \in \{1, \dots, m\}, \forall j \in \{1, \dots, l\},\ \forall z(k) \in \ \mathrm{vert}\left\{\mathfrak{X}^{t+k|t}(x(t))\right\} \tag{34}$$

$$\begin{bmatrix} 1 & x'(t)R_x^{1/2} & \hat{c}'(t|t)R_u^{1/2} \\ (*) & J_0 I & 0 \\ (*) & 0 & J_0 I \end{bmatrix} \geq 0 \tag{35}$$

$$\begin{bmatrix} 1 & z'(k)R_x^{1/2} & \hat{c}'(t+k|t)R_u^{1/2} \\ (*) & J_k I & 0 \\ (*) & 0 & J_k I \end{bmatrix} \geq 0,\ \forall k \in \{1, \dots, N-1\},\ \forall z(k) \in \ \mathrm{vert}\left\{\mathfrak{X}^{t+k|t}(x(t))\right\} \tag{36}$$

2. Feed the plant by the input

$$u(t) = F(t)x(t) + c^*(t|t) \tag{37}$$

3. For any $t > 0$, given $\{F_j(t+N|t)\}_{j=1}^{l}$ and $\mathfrak{X}^{t+N|t}(x(t))$, find

$$\left[\left\{Q_j(t+1), F_j(t+1|t)_{j=1}^{l}, \right\}_{j=1}^{l}, G(t+1), \gamma(t+1)\right] := \arg \min_{Y_j, Q_j>0, \gamma>0} \gamma \tag{38}$$

subject to

- LMIs (20)–(22);
- $$\begin{bmatrix} 1 & ((A_i + B_iF_j)z)^{\mathrm{T}} \\ (A_i + B_iF_j)z & Q_k \end{bmatrix} \geq 0,\ \{i, j, k\} = 1, \dots, l,\ \forall z \in \mathrm{vert}\left\{\mathfrak{X}^{t+N|t}(x(t))\right\} \tag{39}$$
- $$\frac{\gamma}{\gamma(t)} Q_j(t) \leq Q_j \leq Q_j(t), \quad j = 1, \dots, l \tag{40}$$

4. $$\left\{F_i(t+1+k|t+1)\right\}_{i=1}^{l} \leftarrow \left\{F_i(t+k+1|t)\right\}_{i=1}^{l},\ k = 0, \dots, N-1 \tag{41}$$

and compute

$$\left\{F_i(t+1+N|t+1)\right\}_{i=1}^{l} = \left\{Y_i(t+1)Q_i^{-1}(t+1)\right\}_{i=1}^{l}$$

5. $t \leftarrow t+1$ and go to Step 1.

Some comments regarding the structure and the features of the proposed algorithm are mandatory.

Remark 1: The initialisation phase is devoted to ensure that a scheduled stabilising feedback exists which is able to steer asymptotically the initial state x(0) to 0_n without constraints violation. If solvable, *Initialisation* ensures the feasibility of the overall algorithm for all t. Note that such a preliminary step resorts to the one-step ahead state prediction strategy used in [1]. As a consequence, taking into account the LPV hypothesis, the one-step ahead state prediction set reduces to a single vector. This feature allows a start-up procedure which can be potentially less conservative with regard to the standard robust approach of [18].

Remark 2: The terminal set updating procedure (Step 3 of the proposed algorithm), can be regarded as a 'replica' of the initialisation phase with the difference that the joint knowledge of the free input moves $c^*(\cdot|t)$, that are generated during Step 2. and the updated information of the actual state enable to possibly reduce the terminal set 'size' thus enforcing the regulated trajectory to go faster to the desired set point.

Remark 3: The use of a scheduling control law generates an MPC algorithm having a more demanding computational burden with regard to a paradigm which exploits a single feedback gain instead. Anyway, the proposed framework is by no means attractive because of the fact that all features (on-line control moves + time-varying terminal set) are taken into consideration.

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5 Feasibility and stability results

The next proposition ensures that the proposed algorithm DL–LPV–TS admits, at each time instant t, an admissible solution provided that the *Initialisation* is feasible.

Proposition 4: Existence of solutions $(\{Q_j(0), F_j(\cdot|0)\}_{j=1}^{l}, G(0), \gamma(0))$ to the initialisation step implies the existence of solutions $c(\cdot|t)$ of (30) and $(\{Q_j(t), F_j(\cdot|t)\}_{j=1}^{l}, G(t), \gamma(t))$ of (38) for all $t \geq 0$ and one has that

$$0 < Q_j(t+1) \leq Q_j(t), \quad j = 1, \ldots, l$$
$$0 < P_j(t+1) \leq P_j(t), \quad j = 1, \ldots, l$$
$$0 < \gamma(t+1) \leq \gamma(t), \quad \forall t \geq 0 \qquad (42)$$

Proof: Because of the positive invariance of $\mathcal{E}(\{P_j(0)\}_{j=1}^{l}, \gamma(0))$, under the scheduling feedback control law based on the bank of feedback controllers $F_j(1|0)$, $j = 1, \ldots, l$, determined at the initialisation step, the sequence $c(\cdot|t) \equiv 0_m$ is an admissible solution for the generic step at each future time instant t provided that

$$\left\{F_j(t+N|t)\right\}_{j=1}^{l} = \left\{F_j(1|0)\right\}_{j=1}^{l}, \quad \forall t \in \mathbb{Z}_+$$

Assume now that

$$\left(\left\{Q_j(t+1), F_j(t+1|t)\right\}_{j=1}^{l}, G(t+1), \gamma(t+1)\right)$$

is a solution of Step 3 at time t. Conditions (31, 32) ensure that

$$\mathfrak{X}^{t+N|t}(x(t)) \subset \mathcal{E}\left(\left\{P_j(t)\right\}_{j=1}^{l}, \gamma(t)\right)$$

this condition holds because of the fact the J_N denotes an upper-bound to the terminal cost and $\gamma(t)$ is the radius of the terminal ellipsoidal set $\mathcal{E}$ at time t [3, 19]. Moreover, positive invariance of $\mathcal{E}\{(P_j(t)\}_{j=1}^{l}, \gamma(t))$ under any scheduled control law based on $\{F_j(t+N|t)\}_{j=1}^{l}$ and contraction ensures that

$$A_F(\hat{\theta}(t+N|t))\mathfrak{X}^{t+N|t}(x(t)) \subset \mathcal{E}\left(\left\{P_j(t+1)\right\}_{j=1}^{l}, \gamma(t+1)\right),$$
$$\forall \hat{\theta}(t+N|t) \in \Theta^N(t)$$

with

$$Q_j(t+1) \leq Q_j(t), \quad j = 1, \ldots, l$$
$$\mathcal{E}\left(\left\{P_j(t+1)\right\}_{j=1}^{l}, \gamma(t+1)\right) \subset \mathcal{E}\left(\left\{P_j(t)\right\}_{j=1}^{l}, \gamma(t)\right)$$

Finally, given the generic time instant t, Step 3 is always solvable by means of the following, not necessarily optimal choice

$$Q_j(t+1) = Q_j(t), \quad j = 1, \ldots, l, \quad \gamma(t+1) = \gamma(t)$$

which reduces to a 'Frozen'-in-time positive invariant set. □

It remains to show that the proposed MPC scheme guarantees that the resulting closed-loop system is asymptotically stable.

Proposition 5: Let the system (1) be uniformly detectable [20]. Then, the control strategy satisfies the prescribed constraints and yields an asymptotically stable closed-loop system provided the Initialisation step is solvable.

Proof: Let $\{c^*(t+k|t)\}_{k=0}^{N-1}$ be a solution of Step 2. of the proposed algorithm at the generic time t. A feasible input virtual sequence at time $t+1$ is given by

$$c(t+1+k|t+1) \leftarrow c^*(t+k+1|t), k = 0, 1, \ldots, N-2,$$
$$c(t+N-1|t+1) \leftarrow 0_m \qquad (43)$$

Because $c(\cdot|t+1)$ need not to be optimal, it is straightforward to show that

$$V(x(t+1), \left\{P_j(t+1)\right\}_{j=1}^{l}, \left\{F_j(\cdot|t+1)\right\}_{j=1}^{l}, c(\cdot|t+1))$$
$$\leq V(x(t), \left\{P_j(t)\right\}_{j=1}^{l}, \left\{F_j(\cdot|t)\right\}_{j=1}^{l},$$
$$c^*(\cdot|t)) - \|x(t)\|_{R_x}^2 - \left\|c^*(t|t)\right\|_{R_u}^2$$
$$- \max_{z \in \mathrm{vert}\{\mathfrak{X}^{t+N|t}(x(t))\}, P_j(t), j=1,\ldots,l} \|z\|_{P_j(t)}^2$$
$$+ \max_{z \in \mathrm{vert}\{\mathfrak{X}^{t+1+N|t+1}(x(t+1))\}, P_j(t+1), j=1,\ldots,l} \|z\|_{P_j(t+1)}^2. \qquad (44)$$

Hence by the standard Riccati inequality arguments (see [17]) the sum of the last three terms of the r.h.s. of the inequality (44) is negative definite, and consequently we have

$$V(x(t+1), \left\{P_j(t+1)\right\}_{j=1}^{l}, \left\{F_j(\cdot|t+1)\right\}_{j=1}^{l} c(\cdot|t+1))$$
$$- V(x(t), \left\{P_j(t)\right\}_{j=1}^{l}, \left\{F_j(\cdot|t)\right\}_{j=1}^{l}, c^*(\cdot|t))$$
$$\leq -\|x(t)\|_{R_x}^2 - \left\|c^*(t|t)\right\|_{R_u}^2.$$

because $P_j(t) - P_j(t+1) \geq 0, j = 1, \ldots, l, \forall t$.

This means that the sequence $\{V(x(t), \{P_j(t)\}_{j=1}^{l}, \{F_j(\cdot|t)\}_{j=1}^{l}, c^*(\cdot|t))\}_{t=0}^{\infty}$ is monotonically non-increasing and admits a unique limit. Therefore

$$\lim_{t \to +\infty} c^*(t|t) = 0_m$$

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and, as a consequence

$$\sum_{t=0}^{\infty} \|x(t)\|^2_{\mathrm{R}_x} + \|c(t)\|^2_{\mathrm{R}_u} < \infty.$$

Hence, $\lim_{t\to\infty} y(t) = 0_p$, $\lim_{t\to\infty} c(t) = 0_m$ and, by detectability condition, $\lim_{t\to\infty} x(t) = 0_n$. □

6 Illustrative example

The aim of this section is to test the effectiveness of the proposed on-line MPC strategy. To this end, the scheme will be compared in terms of both control performance and computational complexity with the algorithms given in [15] (NDL–LPV), based on standard quadratic stabilisability conditions, and [11] (DL–LPV), a RHC strategy based on dilation techniques. All the computations have been carried out on a Pentium IV using the YALMIP toolbox [21].

The following example is taken from [3] where a robust servomechanism is designed for a polytopic uncertain system subject to a previewable reference signal. In particular, the plant output is required to track an assigned reference trajectory $y_r(t) = C_r x_r(t)$, whose samples are computed from the state of the autonomous system (signal generator)

$$x_r(t+1) = A_r x_r(t), \quad x_r(0) = x_{r0} \tag{45}$$

and a certain number of future samples of the reference signal are available at each time instant for predictions.

The overall control procedure, presented in the previous sections for regulation problems, can also be used to solve tracking problems by resorting to incremental system description and by using a suitably augmented state (see [22] and [23] for details). We consider the following discrete-time LPV system

$$\begin{cases} A(\theta) &= \begin{bmatrix} 0.5 & 1 \\ 0.25 & 1.5 \end{bmatrix}\theta_1 + \begin{bmatrix} 1.5 & 1 \\ 0.75 & 0.2 \end{bmatrix}\theta_2, \\ B(\theta) &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}\theta_1 + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\theta_2 \\ C &= \begin{bmatrix} 0 & 1 \end{bmatrix} \end{cases}$$

with $\theta = [\theta_1, \theta_2]$ in the simplex unit of $\mathbb{R}^2$ having the following structure

$$\theta(t) := \begin{cases} \theta_1(t) = \theta_2(t) = 0.5, \quad 0 \le t < 30\,\mathrm{s} \\ \theta_1(t) = 0.5 + 0.01t, \quad \theta_2(t) = 0.5 - 0.01t, \\ \qquad 30 \le t < 45\,\mathrm{s} \\ \theta_1(t) = 0.8 - 0.01t, \quad \theta_2(t) = 0.2 + 0.01t, \\ \qquad t \ge 45\,\mathrm{s} \end{cases}$$

An incremental augmented model with preview length $h = 2$ has been built up. The state and input weighting matrices has been chosen equal to $\mathrm{R}_x = 1$, $\mathrm{R}_u = \mathrm{diag}(1, 0_{2\times 2}, 0_{2\times 2})$. The reference signal is the following sinusoidal wave $r(t) = \sin((\pi/30)t)$, and the constraint on the control input increments has been chosen equal to $|u(t+1) - u(t)| \le 0.3$.

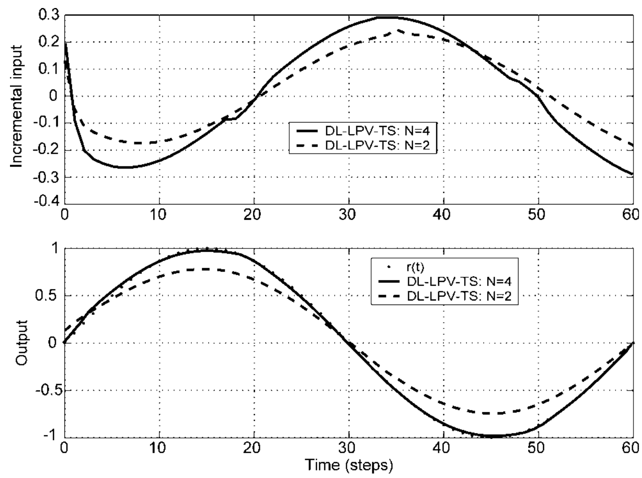

Figure 1 *Tracking output and incremental control*

Algorithm DL–LPV–TS, $N = 2$ represented by dashed line in the graph, $N = 4$ represented by continuous line in the graph

In Fig. 1, for the proposed MPC scheme DL–LPV–TS, the tracking performance in terms of regulated trajectory and incremental input for control horizons $N = 2$, 4 are depicted for a simulation interval of 60 time units. It appears that increasing of N implies a significant improvement in the control performance measured in terms of tracking error and reduction of phase lag.

In Fig. 2, the DL–LPV–TS MPC strategy is compared in terms of regulated responses with respect to its NDL–

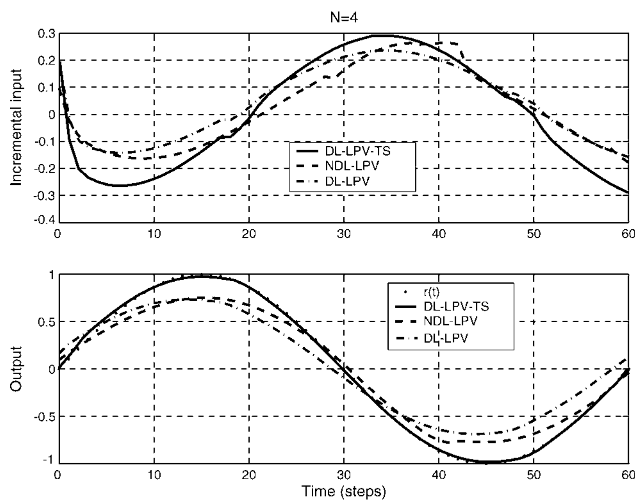

Figure 2 *Tracking output and incremental control*

Comparison between DL–LPV–TS, represented by continuous line in the graph, NDL–LPV, represented by dashed line in the graph, and DL–LPV, represented by dashed-dotted line in the graph, $N = 4$

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Table 1 On-line numerical burdens: average CPU Time (s) per step

Algorithm	Average CPU time
DL−LPV−TS	1.508
NDL−LPV	1.404
DL−LPV	0.436

LPV and DL−LPV counterparts for a control horizon $N = 4$. The simulation parameters are the same of the previous experiment. It can be noted that the proposed dilated scheme exhibits better tracking response and incremental input characteristics both with respect to the one computed by the NDL−LPV algorithm and (more significant) the DL−LPV algorithm.

The computational efforts for the three algorithms are reported in Table 1. As expected, because of the presence of a higher number of constraints, the proposed DL−LPV−TS algorithm shows an on-line worse computational burden than the others. The lightest computational effort is obviously exhibited by the DL−LPV scheme because of the absence of predictions and corresponding free input terms in the control strategy. A slightly lighter (with regard to DL−LPV−TS) computational burden is instead observed for the NDL−LPV paradigm which uses traditional quadratic stability conditions.

7 Conclusion

A dilation-based LPV predictive control strategy has been presented for discrete-time systems described by polytopic (multi-model) representations. The LPV parameter has been assumed measurable and we have considered a stabilising scheduling state-feedback controller. The main contribution of the proposed strategy is to extend the MPC scheme presented in [11] to the general case of control horizons of arbitrary length N.

The improvement accomplished with this MPC scheme with regard to standard LPV algorithms, which make use of quadratic stability arguments [15], relies on the use of PDLFs which enable one to redefine less conservatively the set of all k-step ahead state predictions.

Numerical experiences have shown performance improvements in terms of control input activity and regulated response when the algorithm is compared both to the non-dilated LPV and the RHC LPV schemes.

8 References

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9 Appendix

Proof of Proposition 3: Let us first consider the quadratic stability and robust positive invariance constraints (24), and (25). At each time step t we want to determine sufficient conditions under which the virtual control moves $\hat{u}(t+k|t) = \mathrm{F}(t)\hat{x}(t+k|t)$, $k \geq 0$ robustly stabilise (1) along the state predictions $\hat{x}(t+k|t)$, $k \geq 0$ and minimise an upper bound $\gamma(t)$ to the cost function $J(x(t), u(\cdot|t))$. To this end, we define the quadratic function (see [7, 9])

$$\mathrm{V}(k,t) = \hat{x}(t+k|t)^{\mathrm{T}} P(k,t)\,\hat{x}(t+k|t), \quad \forall k, t \geq 0 \tag{46}$$

with

$$\mathrm{P}(k,t) \triangleq \sum_{j=1}^{l} \hat{\theta}_j(t+k|t)\mathrm{P}_j(t), \quad k \geq 0 \tag{47}$$

In order to satisfy the upper bound (15), the Lyapunov increments must satisfy the inequality

$$V(k+1,t) - V(k,t) \leq -[\hat{x}^{\mathrm{T}}(t+k|t)\mathrm{R}_x\hat{x}(t+k|t) + u^{\mathrm{T}}(t+k|t)\mathrm{R}_u u(t+k|t)], \quad \forall \hat{\theta}(t+k|t) \in \Theta^*, \quad k \geq 0 \tag{48}$$

The $k+1$ step ahead state predictions, under the control strategy (23), is given by $\hat{x}(t+k+1|t) = A_F(k,t)\hat{x}(t+k|t)$. Then, (48) can be rewritten as

$$\hat{x}^{\mathrm{T}}(t+k|t)[A_F^{\mathrm{T}}(k,t)\,\mathrm{P}(k+1,t)\mathrm{A}_F(k,t) - \mathrm{P}(k,t) + \mathrm{F}(t)^{\mathrm{T}}\mathrm{R}_u\mathrm{F}(t) + \mathrm{R}_x]\hat{x}(t+k|t) \leq 0 \tag{49}$$

with

$$\mathrm{A_F}(k,t) \triangleq \sum_{i=1}^{l}\sum_{j=1}^{l} \hat{\theta}_i(t+k|t)\hat{\theta}_j(t+k|t)\Big(A_i + B_i\mathrm{F}_j(t)\Big) \tag{50}$$

$$\mathrm{F}(t) \triangleq \sum_{i=1}^{l} \hat{\theta}_i(t|t)\mathrm{F}_j(t) \tag{51}$$

$$\mathrm{P}(k,t) \triangleq \sum_{i=1}^{l} \hat{\theta}_i(t+k|t)\mathrm{P}_i(t), \quad k \geq 0 \tag{52}$$

$$\mathrm{P}(k+1,t) \triangleq \sum_{s=1}^{l} \theta_s^{+}(t+k|t)P_s(t), \quad k \geq 0 \tag{53}$$

where $\hat{\theta}_s^{+}(t+k|t) \triangleq \hat{\theta}_s(t+k+1|t)$ denotes the one-step ahead variation of the scheduling parameter prediction $\hat{\theta}(\cdot|t)$. By means of Schur complements, (49) is satisfied if the following matrix inequality holds

$$\begin{bmatrix} \mathrm{P}(k,t) & * & * & * \\ \mathrm{P}(k+1,t)\mathrm{P_F}(k,t) & \mathrm{P}(k+1,t) & * & * \\ \mathrm{R}_x^{1/2} & 0 & \mathrm{I} & * \\ \mathrm{R}_u^{1/2}\mathrm{F}(t) & 0 & 0 & \mathrm{I} \end{bmatrix} \geq 0, \quad \forall t, k \geq 0 \tag{54}$$

In turn, inequality (54) holds true if

$$\begin{bmatrix} \mathrm{P}(k,t) & * & * & * \\ \mathrm{P}_s\mathrm{A_F}(k,t) & \mathrm{P}_s & * & * \\ \mathrm{R}_x^{1/2} & 0 & \mathrm{I} & * \\ \mathrm{R}_u^{1/2}F(t) & 0 & 0 & \mathrm{I} \end{bmatrix} \geq 0 \quad s = 1, \ldots, l \tag{55}$$

is satisfied over the vertices of $\mathrm{P}(k+1,t)$ (53). Then, by denoting $\mathrm{Q}_s = \gamma\mathrm{P}_s^{-1}$ and applying the congruence transformation $\mathrm{diag}(\gamma^{-1/2}\mathrm{G}^{\mathrm{T}}, \gamma^{-1/2}\mathrm{Q}_s, I, I)$ to (55), one

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obtains that matrix inequality (54) is satisfied if

$$\begin{bmatrix} \gamma^{-1}\mathrm{G}^{\mathrm{T}}\mathrm{P}_i\mathrm{G} & * & * & * \\ (\mathrm{A}_i+\mathrm{B}_i\mathrm{F}_j)\mathrm{G} & \mathrm{Q}_s & * & * \\ \gamma^{-1/2}\mathrm{R}_x^{1/2}\mathrm{G} & 0 & \mathrm{I} & * \\ \gamma^{-1/2}\mathrm{R}_u^{1/2}\mathrm{F}_j\mathrm{G} & 0 & 0 & \mathrm{I} \end{bmatrix} \geq 0 \quad i,j,s=1,\ldots,l \tag{56}$$

The (l^3) sufficient conditions for the satisfaction of (56) can be reduced by rewriting the one-step ahead closed loop system predictions, starting from $\hat{x}(t+k|t)$, as

$$\hat{x}(t+k+1|t)=\sum_{i=1}^{l}\sum_{j=1}^{l}\hat{\theta}_i(t+k|t)\hat{\theta}_j(t+k|t)$$

$$(A_i+B_iF_j)\,\hat{x}(t+k|t)=\sum_{i=1}^{l}\hat{\theta}_i^2(t+k|t)(A_i+B_iF_i)\,\hat{x}(t+k|t)$$

$$+2\sum_{i=1}^{l}\sum_{j>i}\hat{\theta}_i(t+k|t)\hat{\theta}_j(t+k|t)$$

$$\times\frac{(A_i+B_iF_j)+(A_j+B_jF_i)}{2}\hat{x}(t+k|t)$$

with the vector $[\hat{\theta}_1^2,\ldots,\hat{\theta}_l^2, 2\,\hat{\theta}_1\,\hat{\theta}_2,\ldots,2\,\hat{\theta}_{l-1}\,\hat{\theta}_l]$ belonging to the unit simplex

$$\sum_{i=1}^{l}\hat{\theta}_i^2(t+k|t)+2\sum_{i=1}^{l}\sum_{j>i}\hat{\theta}_i(t+k|t)\hat{\theta}_j(t+k|t)=1 \tag{57}$$

As a consequence in (50), for a given time instants couple (t,k), it is possible to rewrite $A_F(k,t)$ as a convex combination of 'half-sum' terms

$$\begin{aligned}\mathrm{A_F}(k,t)&=\sum_{i=1}^{l}\hat{\theta}_i^2(t+k|t)(\mathrm{A}_i+\mathrm{B}_i\mathrm{F}_i)\\&+\sum_{i=1}^{l}\sum_{j>i}2\hat{\theta}_i(t+k|t)\hat{\theta}_j(t+k|t)\\&\times\left(\frac{\mathrm{A}_i+\mathrm{A}_j+\mathrm{B}_j\mathrm{F}_i(t)+\mathrm{B}_i\mathrm{F}_j(t)}{2}\right)\end{aligned} \tag{58}$$

The feedback gain F(t) and the Lyapunov matrix P(k,t) can be recast in a similar way as (58)

$$\begin{aligned}F(t)&=\sum_{i=1}^{l}\hat{\theta}_i^2(t+k|t)F_i(t)\\&+\sum_{i=1}^{l}\sum_{j>i}2\hat{\theta}_i(t|t)\hat{\theta}_j(t|t)\left(\frac{F_i(t)+F_j(t)}{2}\right)\end{aligned} \tag{59}$$

$$\begin{aligned}\mathrm{P}(k,t)&=\sum_{i=1}^{l}\hat{\theta}_i^2(t+k|t)\mathrm{P}_i(t)\\&+\sum_{i=1}^{l}\sum_{j>i}2\hat{\theta}_i(t+k|t)\,\hat{\theta}_j(t+k|t)\left(\frac{\mathrm{P}_i(t)+\mathrm{P}_j(t)}{2}\right)\end{aligned} \tag{60}$$

Thank to the above rewriting, sufficient conditions guaranteeing the validity of (56), are given by

$$\begin{bmatrix} \gamma^{-1}\left(\frac{\mathrm{G}^{\mathrm{T}}\mathrm{P}_i\mathrm{G}+\mathrm{G}^{\mathrm{T}}\mathrm{P}_j\mathrm{G}}{2}\right) & * & * & * \\ \left(\frac{\mathrm{A}_i+\mathrm{A}_j+\mathrm{B}_i\mathrm{F}_j+\mathrm{B}_j\mathrm{F}_i}{2}\right)\mathrm{G} & \mathrm{Q}_s & * & * \\ \gamma^{-1/2}\mathrm{R}_x^{1/2}\mathrm{G} & 0 & 1I & * \\ \gamma^{-1/2}\mathrm{R}_u^{1/2}\left(\frac{\mathrm{F}_i+\mathrm{F}_j}{2}\right)\mathrm{G} & 0 & 0 & I \end{bmatrix} \geq 0,\, i,j,k=1,\ldots,l \tag{61}$$

By applying the *dilation inequality* (see [8]), namely

$$\mathrm{G}^{\mathrm{T}}\mathrm{S}^{-1}\mathrm{G}\geq\mathrm{G}+\mathrm{G}^{\mathrm{T}}-\mathrm{S}$$

to the (1,1) block of (61), the condition (20) results via the standard variable change $F_i=\mathrm{Y}_i\mathrm{G}^{-1}$. The invariance condition (21) can be trivially proved by the same arguments used in [18]. Let us now consider the input constraint condition

$$|\hat{u}_j(t+k|t)|\leq u_{j,\max},\; k\geq 0,\quad j=1,\ldots,m \tag{62}$$

Using a series of well-known technicalities (see [18]), (62) is satisfied if

$$\begin{aligned}\max_{k\geq 0}\left|\hat{u}_j(t+k|t)\right|^2&\leq\max_{k\geq 0}\left|(F(t)\hat{x}(t+k|t))_j\right|^2\\&\leq\max_{z\in\mathcal{E}}\left|(F(t)z)_j\right|^2\end{aligned} \tag{63}$$

Introducing $W(k,t)=\gamma P^{-1}(k,t)$, one has

$$\max_{z\in\mathcal{E}}\left|(\mathrm{F}(t)z)_j\right|^2=\max_{z\in\mathcal{E}}\left|\left(\mathrm{F}(t)W^{1/2}(k,t)W^{-1/2}(k,t)z\right)_j\right|^2$$

which implies

$$\begin{aligned}&\max_{z\in\mathcal{E}}\left|\left(\mathrm{F}(t)W^{-1/2}(k,t)W^{-1/2}(k,t)z\right)_j\right|^2\\&\leq\left\|\left(\mathrm{F}(t)W(k,t)^{1/2}\right)_j\right\|_2^2\\&=\left(\mathrm{F}(t)W(k,t)\mathrm{F}^{\mathrm{T}}(t)\right)_{jj}\leq u_{j,\max}^2\end{aligned}$$

By imposing the existence of a matrix $Z = Z^T > 0$ such that

$$(Z)_{kk} \leq u_{k,\max}^2, k = 1, \ldots, m \tag{64}$$

we obtain that (62) is implied by

$$\begin{bmatrix} Z & F(t) \\ F^T(t) & W^{-1}(t,k) \end{bmatrix} \geq 0 \tag{65}$$

Finally, by applying the congruence transformation diag(I, G) to (65) and recalling the expressions of F(t) and P(k,t) (23)–(47), (65) becomes

$$\begin{bmatrix} Z & Y_j \\ Y_j^T & G^T Q_j^{-1} G \end{bmatrix} \geq 0, \quad j = 1, \ldots, l \tag{66}$$

and easily dilated as in (22).

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